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STABILITY AND CONTROL. VOLUME II. STABILITY AND  
CONTROL FLIGHT TEST THEORY

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Edwards Air Force Base, California

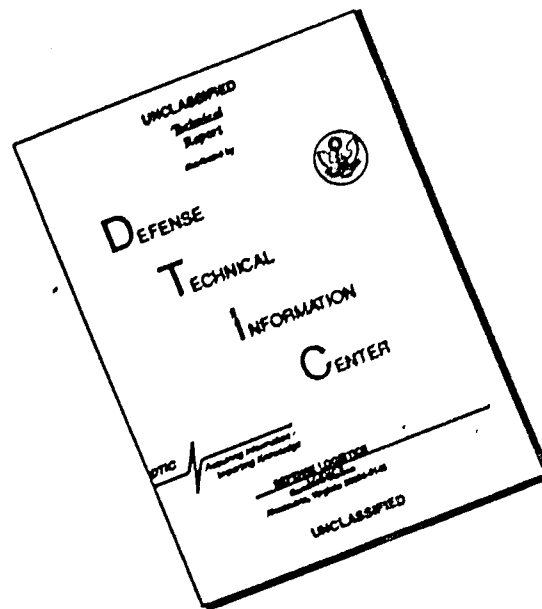
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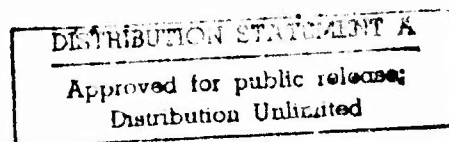


STABILITY AND CONTROL  
Volume II of II  
Stability and Control Flight Test Theory

July 1974

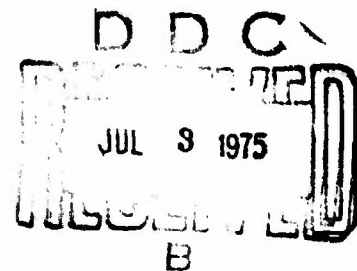
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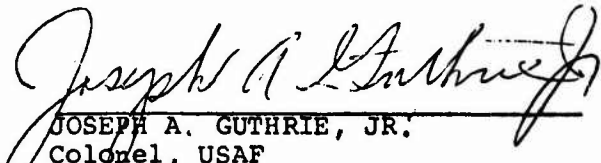
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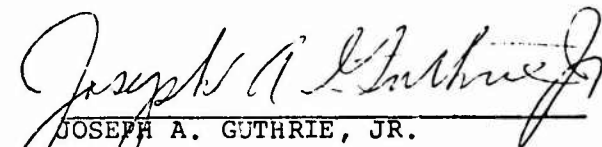
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## PREFACE

Stability and Control is that branch of the aeronautical sciences that is concerned with giving the pilot an aircraft with good handling qualities. As aircraft have been designed to meet greater performance specifications, new problems in Stability and Control have been encountered. The solving of these problems has advanced the science of Stability and Control to the point it is today.

This handbook has been compiled by the instructors of the USAF Test Pilot School for use in the Stability and Control portion of the School's course. Most of the material in Volume I of this handbook has been extracted from several reference books and is oriented towards the test pilot. The flight test techniques and data reduction methods in Volume II have been developed at the Air Force Flight Test Center, Edwards Air Force Base, California. This handbook is primarily intended to be used as an academic text in our School, but if it can be helpful to anyone in the conduct of Stability and Control testing, be our guest.

  
JOSEPH A. GUTHRIE, JR.  
Colonel, USAF  
Commandant, USAF Test Pilot School

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**DIFFERENTIAL EQUATIONS**

(REVISED MAY 1975)

**list of abbreviations and symbols**

<u>Item</u>	<u>Definition</u>
$x, y, z$	variables
$t$	time in seconds
$p$	differential operator with dimensions of (seconds) <sup>-1</sup>
$j$	constant equal to $\sqrt{-1}$
$\phi, \theta$	angular constant in radians
$e$	constant equal to $\lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = 2.71828. . . . .$
$x_t, y_t, z_t$	transient solution to differential equation
$x_p, y_p, z_p$	particular (steady state) solution to differential equation
$\dot{x}$	the dot notation indicates differentiation with respect to time, as in $\dot{x} = \frac{dx}{dt}$
$\tau$	time constant in seconds
$T_1$	time to half amplitude in seconds
$\zeta$	damping ratio
$\omega_n$	undamped natural frequency in radians per second
$\omega_d$	damped frequency in radians per second
$s$	Laplace variable with dimensions of (seconds) <sup>-1</sup>
$L$	Laplace transform
$L^{-1}$	inverse Laplace transform
$X(s), Y(s), Z(s)$	Laplace transform of $x(t), y(t), z(t)$
$\Delta$ =	symbol used for definitions, such as $\dot{x} \triangleq \frac{dx}{dt}$ means $\dot{x}$ is defined as $\frac{dx}{dt}$

## 1.1 INTRODUCTION

The theory of differential equations is a subject of considerable scope, ranging from the rather simple and obvious through the abstract and not so obvious. One can spend a lifetime studying the subject, and a few people have. We have neither the time, nor perhaps the inclination for such devotions. Our purpose is to cover those aspects of the theory of differential equations which are of direct application to work at the school.

These notes deal with the tools and techniques required to analyze differential equations. Such techniques are easily extended for use in the study of aircraft dynamics. An aircraft in flight displays motions similar to a mass-spring-damper system (figure 1.1). The static stability of the airplane is similar to the spring, the moments of inertia similar to the mass, and the airflow serves to damp the aircraft motion.

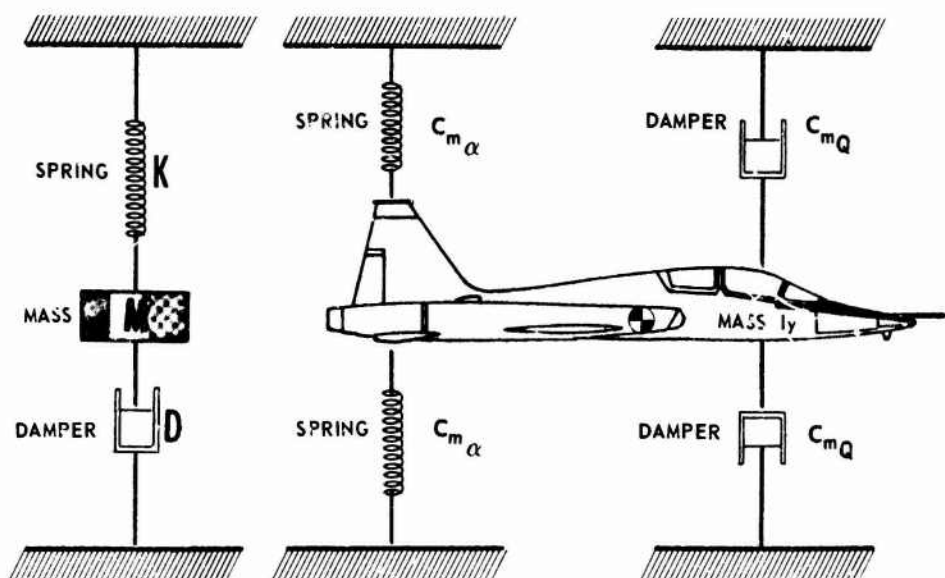


Figure 1.1

This first section provides a review of basic differential equation theory. Succeeding sections deal with operator techniques, analysis of first and second order systems, use of Laplace transforms, and solution of simultaneous equations.

Before proceeding with our study, we shall define several terms which will be used in these notes.

Differential Equation - An equation which involves a dependent variable (or variables) together with one or more of its derivatives with respect to an independent variable (or variables).

Solution - Any function, free of derivatives, which satisfies a differential equation is said to be a solution of the differential equation.

Ordinary Differential Equation - A differential equation which involves derivatives with respect to a single independent variable is called an ordinary differential equation.

Order - The  $n^{\text{th}}$  derivative of a dependent variable is called a derivative of order  $n$ , or an  $n^{\text{th}}$  order derivative. The order of a differential equation is the order of the highest order derivative present.

Degree - The exponent of the highest order derivative is called the degree of the differential equation.

Linear Differential Equation (ordinary, single dependent variable) - A differential equation in which the dependent variable and its derivatives appear in no higher than the 1st degree, and the coefficients are either constants or functions of the independent variable, is called a linear differential equation.

Linear System - Any physical system that can be described by a linear differential equation is called a linear system.

General Solution - A solution of a differential equation of order  $n$  which contains  $n$  arbitrary constants will be called a general solution of the differential equation.

## ■ 1.2 REVIEW OF BASIC PRINCIPLES

Before investigating operator notation and Laplace transforms, let's review the more basic methods of solving differential equations.

### ● 1.2.1 DIRECT INTEGRATION

To solve a differential equation we seek a mathematical expression, relating the variables appearing in the differential equation, which qualifies as a solution under the definitions given above. A first thought or inspiration may be: since we are presented with an equation containing derivatives, a solution may be obtained by antidifferentiating or integration. This process removes derivatives and provides arbitrary constants.

#### EXAMPLE

Given

$$\frac{dy}{dx} = x + 4$$

rewriting

$$dy = (x + 4)dx$$

integrating

$$dy = (x + 4)dx + C$$

gives us

$$y = \frac{x^2}{2} + 4x + C \quad (1.1)$$

#### EXAMPLE

Given

$$\frac{d^2y}{dx^2} = x + 4$$

Assume

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx}$$

where

$$y' = \frac{dy}{dx}$$

then

$$\frac{d(y')}{dx} = x + 4$$

$$\text{or } d(y') = (x + 4)dx + C_1$$

then

$$y' = \frac{dy}{dx} = \frac{x^2}{2} + 4x + C_1$$

integrating again

$$dy = \left(\frac{x^2}{2} + 4x + C_1\right) dx + C_2$$

giving

$$y = \frac{x^3}{6} + 2x^2 + C_1x + C_2 \quad (1.2)$$

Equations 1.1 and 1.2 qualify as general solutions under the definition stated earlier.

Life is full of disappointments and we would soon learn that this direct application of the integration process would fail to work in many cases.

#### EXAMPLE

$$2xy + (x^2 + \cos y) \frac{dy}{dx} = 0 \quad (1.3)$$

or

$$\begin{aligned} dy &= \frac{-2xy}{x^2 + \cos y} \\ dy &= \frac{-2xy}{x^2 + \cos y} dx + C \end{aligned} \quad (1.4)$$

We cannot perform the integration of the term to the right of the equal sign in equation 1.4. Equation 1.3 can be solved, however, using straightforward techniques. ( $x^2 + \sin y = c$  is a general solution.) We emphasize the word "technique" since the solution may rely upon novel approaches, special groupings, or "judicious arrangements" and, perhaps, witchcraft or conjuring. The former require extensive experience and maturity within the discipline, and the latter talents are rarely endowed by nature. We shall study a few special differential equations which are easy to solve and have wide application in the analysis of physical problems.

#### 1.2.2 FIRST ORDER EQUATIONS

We shall consider briefly the first order ordinary differential equation. Suppose we represent such an equation by

$$F(y', y, x) = 0$$

where

$$y' = \frac{dy}{dx}$$

This is concise notation used by mathematicians to denote a differential equation containing an independent variable  $x$ , a dependent variable  $y$ , and the derivative of  $y$  with respect to  $x$ . The equation may contain the derivative in differential form.

#### EXAMPLES

$$\frac{dy}{dx} = x + y$$

$$3x \, dx + 4y \, dy = 0$$

$$y' = \frac{x - y}{x + y}$$

$$\frac{dy}{dx} = \frac{x - y \cos x}{\sin x + y}$$

First order differential equations may be solved by

1. Separating variables and integrating directly.
2. Recognizing exact forms and integrating directly.
3. Finding an integrating factor (fudge factor) which will make the equation exact.
4. Inspection, rearrangement of terms, etc., to use method 1 or 2, or a combination of the two.

These methods are thoroughly treated in all elementary differential equations texts. A brief review of methods 1 and 2 is given below.

#### ●1.2.2.1 SEPARATION OF VARIABLES

When a differential equation can be put in the form

$$f_1(x)dx + f_2(y)dy = 0 \quad (1.5)$$

where one term contains functions of  $x$  and  $dx$  only, and the other functions of  $y$  and  $dy$  only, the variables are said to be separated. A solution of equation 1.5 can then be obtained by direct integration

$$f_1(x)dx + f_2(y)dy = C \quad (1.6)$$

where  $C$  is an arbitrary constant. Note, that for a differential equation of the first order there is one arbitrary constant. In general, the number of arbitrary constants is equal to the order of the differential equation.

#### EXAMPLE

$$\frac{dy}{dx} = \frac{x^2 + 3x + 4}{y + 6}$$

$$(y + 6) \, dy = (x^2 + 3x + 4) \, dx$$

$$(y + 6) \, dy = (x^2 + 3x + 4) \, dx + C$$

$$\frac{y^2}{2} + 6y = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + C$$

### ●1.2.2.2 EXACT DIFFERENTIAL EQUATION

Associated with each suitably differentiable function of two variables  $f(x,y)$  there is an expression called its total differential, namely,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad (1.7)$$

Conversely, if the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \quad (1.8)$$

has the property that

$$M(x,y) = \frac{\partial f}{\partial x} \text{ and } N(x,y) = \frac{\partial f}{\partial y}$$

then it can be rewritten in the form

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df = 0$$

from which it follows that

$$f(x,y) = C$$

is a solution. Equations of this sort are said to be exact, since, as they stand, their left members are exact differentials.

A differential equation

$$M(x,y) dx + N(x,y) dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (1.9)$$

If the differential equation

$$M(x,y) dx + N(x,y) dy = 0$$

is exact, then for all values of  $k$ ,

$$\int_a^x M(x,y) dx + \int_b^y N(a,y) dy = k \quad (1.10)$$

is a solution of the equation.

### EXAMPLE

Show that the equation

$$(2x + 3y - 2)dx + (3x - 4y + 1)dy = 0$$

is exact and find a general solution.

Applying the test, we find

$$\frac{\partial M}{\partial y} = \frac{\partial (2x + 3y - 2)}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = \frac{\partial (3x - 4y + 1)}{\partial x} = 3$$

Since the two partial derivatives are equal, the equation is exact. Its solution can be found by means of equation 1.10.

$$\int_a^x (2x + 3y - 2)dx + \int_b^y (3a - 4y + 1)dy = k$$

$$(x^2 + 3xy - 2x) \Big|_a^x + (3ay - 2y^2 + y) \Big|_b^y = k$$

$$(x^2 + 3xy - 2x) - (a^2 + 3ay - 2a) + (3ay - 2y^2 + y) - (3ab - 2b^2 + b) = k$$

$$x^2 + 3xy - 2x - 2y^2 + y = k + a^2 - 2a + 3ab - 2b^2 + b = K$$

### ●1.2.2.3 FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

We conclude the discussion of first order equations by considering the following form

$$\frac{dy}{dx} + R(x) y = 0 \quad (1.11)$$

where  $R(x)$  may be a constant. To solve, merely separate variables.

$$\frac{dy}{y} + R(x) dx = 0$$

integrating

$$\int \frac{dy}{y} = - \int R(x) dx + C'$$

where

$$C' = \ln C$$

Thus

$$\ln y = - \int R(x) dx + \ln C$$

or

$$y = Ce^{-\int R(x) dx}$$

If  $R$  is a constant, then

$$y = Ce^{-Rx} \quad (1.12)$$

We might conclude from this result that a first order differential equation of form 1.11 with constant coefficients may be solved quite simply. This is true and the solution will always have the form of equation 1.12.

EXAMPLE

$$\frac{dy}{dx} + 2y = 0 \quad (1.13)$$

then we have directly

$$y = Ce^{-2x} \quad (1.14)$$

which is the general solution. It is quickly recognized that the solution is easily obtained by plugging the negative of the coefficient of  $y$  into the position indicated by the small square.

PROBLEMS: Set I, Nos. 1 and 2, page 1.76.

### 1.3 LINEAR DIFFERENTIAL EQUATIONS AND OPERATOR TECHNIQUES

A form of the differential equation that is of particular interest is

$$A_n \frac{d^n y}{dx^n} + A_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + A_1 \frac{dy}{dx} + A_0 y = f(x) \quad (1.15)$$

If the coefficient expressions  $A_n, A_{n-1}, \dots, A_0$  are all functions of  $x$  only, then equation 1.15 is called a linear differential equation. If the coefficient expressions  $A_n, \dots, A_0$  are all constants, then 1.15 is called a linear differential equation with constant coefficients.

EXAMPLE

$$x^2 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + xy = \sin x$$

is a linear differential equation.

EXAMPLE

$$\frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 9y = e^x$$

is a linear differential equation with constant coefficients. Linear differential equations with constant coefficients occur frequently in the analysis of physical systems. Mathematicians and engineers have developed simple and effective techniques to solve this type of equation by using either "classical" or operational methods. When attempting to solve a linear differential equation of the form

$$A_n \frac{d^n y}{dx^n} + A_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + A_1 \frac{dy}{dx} + A_0 y = f(x) \quad (1.16)$$

it is helpful to examine the equation

$$A_n \frac{d^n y}{dx^n} + A_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + A_1 \frac{dy}{dx} + A_0 y = 0 \quad (1.17)$$

1.17 is the same as 1.16 with the right hand side zero. We shall refer to 1.16 as the general equation and equation 1.17 as the complementary or homogeneous equation. Solutions of equation 1.17 possess a useful property known as superposition, which may be briefly stated as follows: Suppose  $y_1(x)$  and  $y_2(x)$  are distinct solutions of 1.17. Then any linear combination of  $y_1(x)$  and  $y_2(x)$  is also a solution of 1.17. A linear combination would be  $C_1 y_1(x) + C_2 y_2(x)$ .

EXAMPLE

$$\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$$

It can be verified that  $y_1(x) = e^{3x}$  is a solution, and that  $y_2(x) = e^{2x}$  is another solution which is distinct from  $y_1(x)$ . Using superposition, then,  $y(x) = c_1 e^{3x} + c_2 e^{2x}$  is also a solution.

Equation 1.16 may be interpreted as representing a physical system where the left side of the equation describes the natural or designed state of the system, and where the right side of the equation represents the input or forcing function.

One might logically pursue the following line of reasoning in attempting to find a solution to the problem described by equation 1.16.

1. A general solution of 1.16 must contain  $n$  arbitrary constants and must satisfy the equation.
2. The following statements are justified by experience:
  - a. It is reasonably straightforward to find a solution to the complementary equation 1.17, containing  $n$  arbitrary constants. Such a solution will be called the transient solution. Physically, it represents the response present in the system regardless of input.
  - b. There are varied techniques for finding a solution of the differential equations due to this forcing function. Such solutions do not, in general, contain arbitrary constants. This solution will be called the particular or steady state solution.
3. If we take the transient solution which describes the response already existing in the system, and then add on the response due to the forcing function, it would appear that a solution so written would blend the two responses and describe the total response of the system represented by 1.16. In fact, the definition of a general solution is satisfied under such an arrangement. This is simply an extension of the principle of superposition. The transient solution contains the correct number of arbitrary constants, and the particular solution guarantees that the combined solutions satisfy the general equation 1.16. Call the transient solution  $y_t$  and the particular solution  $y_p$ . A general solution of 1.16 is then given by

$$y = y_t + y_p \quad (1.18)$$

### ●1.3.1 TRANSIENT SOLUTION

Equation 1.13 is a complementary or homogeneous first order linear differential equation with constant coefficients. We recognized a quick and simple method of finding a solution to this equation. We also recognized that the solution was always of exponential form. We might hope that solutions of higher order equations of the same family would take the same form.

Let us examine a second order differential equation with constant coefficients to determine if

$$y = e^{mx} \quad (1.19)$$

is a solution of the equation

$$ay'' + by' + cy = 0 \quad (1.20)$$

Substituting  $y = e^{mx}$  we have

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

or

$$(am^2 + bm + c)e^{mx} = 0 \quad (1.21)$$

Since  $e^{mx} \neq 0$

$$am^2 + bm + c = 0 \quad (1.22)$$

and

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (1.23)$$

Substituting these values into our assumed solution we force it to become a solution.

$$y_t = C_1 e^{m_1 x} + C_2 e^{m_2 x} \quad (1.24)$$

When working numerical problems it is not necessary to take the derivatives of  $e^{mx}$ , if we remember that the  $d^n y/dx^n$  is replaced by  $m^n$ . This will be true for any order differential equation with constant coefficients.

We have included the subscript "t" on y to indicate that 1.24 represents the transient solution. From the foregoing it is seen we have succeeded in extending the method for first order complementary equations to higher order complementary or homogeneous equations. Again we note that we have traded off an integration problem for an algebra problem (solving equation 1.22 for the m's).

Differential or derivative operators can be defined and manipulated to play the same role as m above.

If we designate an operator p,  $p^2$ , . . . ,  $p^n$  as follows:

$$p = \frac{d}{dx}, \quad p^2 = \frac{d^2}{dx^2}, \quad \dots, \quad p^n = \frac{d^n}{dx^n} \quad (1.25)$$

$$p(y) = \frac{dy}{dx}, \quad p^2(y) = \frac{d^2 y}{dx^2}, \quad \dots, \quad p^n(y) = \frac{d^n y}{dx^n} \quad (1.26)$$

then 1.20 may be written

$$ap^2(y) + bp(y) + cy = 0 \quad (1.27)$$

or, since the derivative operates linearly (each term in succession),

$$(ap^2 + bp + c)y = 0 \quad (1.28)$$

and the operator expression  $(ap^2 + bp + c)$  has the same algebraic structure as 1.22. The operator expression in 1.28 is a polynomial with precisely the same form as the polynomial on the left side of 1.22, hence it is often solved directly for the constants required in the solution of 1.19. In this case, the transient solution 1.24 would appear

$$y_t = c_1 e^{p_1 x} + c_2 e^{p_2 x} \quad (1.29)$$

There are cases for which 1.24 and 1.29 are not entirely satisfactory in providing a solution, but this will be discussed later. The  $m$ 's or  $p$ 's may be real, imaginary, or complex numbers.

#### EXAMPLE

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = 0$$

Using operator notation,

$$(p^2 + p - 2)y = 0$$

$$p^2 + p - 2 = 0$$

$$p = 1, -2$$

$$y = c_1 e^x + c_2 e^{-2x}$$

We shall now consider the various cases for solutions of the complementary (homogeneous) equation.

Consider the equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad (1.30)$$

We have seen above that the solution of this differential equation is equivalent to solving the characteristic equation

$$ap^2 + bp + c = 0 \quad (1.31)$$

The general solution of 1.30 is of the form

$$x_t = c_1 e^{p_1 x} + c_2 e^{p_2 x} \quad (1.32)$$

where  $c_1$  and  $c_2$  are arbitrary constants, and  $p_1$  and  $p_2$  are solutions of the characteristic equation 1.31. Recall from algebra that a characteristic equation can yield complex roots, imaginary roots, or real roots,

that is,  $p_{1,2} = [-b \pm \sqrt{b^2 - 4ac}]/2a$ . We will consider the solution 1.32 for various values of the constants in equation 1.31 and consider changes in the form of the solution which may be desirable or necessary.

#### ●1.3.1.1 CASE 1: ROOTS REAL AND UNEQUAL

If  $p_1$  and  $p_2$  are real and unequal the desired form of solution is just as is

EXAMPLE

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 12y = 0$$

$$(p^2 + 4p - 12)y = 0 \text{ (in operator form)}$$

solving

$$p^2 + 4p - 12 = 0$$

gives

$$p = \frac{-4 \pm \sqrt{16 + 48}}{2}$$

$$= \frac{-4 \pm 8}{2}$$

or

$$p = -6, 2$$

and

$$y = c_1 e^{-6x} + c_2 e^{2x}$$

is the required solution.

●1.3.1.2 CASE 2: ROOTS REAL AND EQUAL

If  $p_1$  and  $p_2$  are real and equal we run into trouble.

EXAMPLE

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$$

$$(p^2 - 4p + 4)y = 0 \text{ (in operator form)}$$

solving,

$$p = \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4}{2} = 2$$

or  $p = 2$ . But this gives only one value of  $p$ . If we try to use 1.32 all we get is  $y = c_1 e^{2x}$  but we need two arbitrary constants to have a transient solution like 1.30. If we are really alert, we may notice that the operator expression  $(p^2 - 4p + 4)$  can be written  $(p - 2)(p - 2)$ , or  $(p - 2)^2$ , which is a polynomial expression with a repeated factor. (that is,  $p = 2$ ; 2 is the solution.) We can then write  $y = c_1 e^{2x} + c_2 x e^{2x}$  as the transient solution. This is really no better than our first attempt,  $y = c_1 e^{2x}$ , since  $c_1$  and  $c_2$  can be combined into a single arbitrary constant.

$$y = c_1 e^{2x} + c_2 x e^{2x} = (c_1 + c_2 x) e^{2x} = c_3 e^{2x}$$

To solve this problem, simply multiply one of the arbitrary constants by  $x$ . Now write:  $y = c_1 e^{2x} + c_2 x e^{2x}$ . We can no longer "lump" the two coefficients of  $e^{2x}$  together. The solution now contains two arbitrary constants, and it is easily verified that

$$y_t = c_1 e^{2x} + c_2 x e^{2x}$$

is a transient solution of the problem above.

●1.3.1.3 CASE 3: ROOTS PURELY IMAGINARY

EXAMPLE

$$\frac{d^2y}{dx^2} + y = 0$$

in operator form

$$(p^2 + 1)y = 0$$

Solving,

$$p = \frac{0 \pm \sqrt{0 - 4}}{2} = \pm \sqrt{-1}$$

In most engineering work we refer to  $\sqrt{-1}$  as  $j$ . (In mathematical texts it is denoted by  $i$ .) Now,

$$p = \pm j$$

and the solution is written

$$y_t = c_1 e^{jx} + c_2 e^{-jx} \quad (1.33)$$

This is a perfectly good solution from a mathematical standpoint, but it is unwieldy and unsuggestive to engineers. A mathematician by the name of Euler worked out this puzzle for us by developing an equation called Euler's identity.

$$e^{jx} = \cos x + j \sin x \quad (1.34)$$

This equation can be restated in many ways geometrically and analytically, and can be verified by adding the series expansion of  $\cos x$  to the series expansion of  $j \sin x$ . Now 1.33 may be expressed

$$\begin{aligned} y_t &= c_1 (\cos x + j \sin x) + c_2 [\cos (-x) + j \sin (-x)] \\ &= (c_1 + c_2) \cos x + j (c_1 - c_2) \sin x \end{aligned} \quad (1.35)$$

or

$$y_t = c_3 \cos x + c_4 \sin x \quad (1.36)$$

Equation 1.36 has another interesting form. Let

$$y_t = \sqrt{c_3^2 + c_4^2} \left[ \frac{c_3}{\sqrt{c_3^2 + c_4^2}} \cos x + \frac{c_4}{\sqrt{c_3^2 + c_4^2}} \sin x \right] \quad (1.37)$$

Now consider a right triangle with sides labeled as follows:

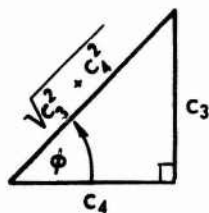


Figure 1.2

Now,

$$\frac{c_3}{\sqrt{c_3^2 + c_4^2}} = \sin \phi$$

$$\frac{c_4}{\sqrt{c_3^2 + c_4^2}} = \cos \phi$$

and

$$\sqrt{c_3^2 + c_4^2} = A.$$

$A$  and  $\phi$  are arbitrary constants, and 1.37 becomes

$$y_t = A (\sin \phi \cos x + \cos \phi \sin x)$$

or

$$y_t = A \sin (x + \phi) \quad (1.38)$$

To summarize, if the roots of the operator polynomial are purely imaginary, they will be numerically equal but opposite in sign, and the solution will have the form 1.36 or 1.38.

#### ●1.3.1.4 CASE 4: ROOTS COMPLEX

EXAMPLE

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0$$

in operator form,

$$(p^2 + 2p + 2)y = 0$$

Solving,

$$p = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1}$$

or

$$p = -1 + j, -1 - j$$

and

$$y_t = c_1 e^{(-1 + j)x} + c_2 e^{(-1 - j)x} \quad (1.39)$$

Equation 1.39 may be written

$$y_t = e^{-x} \left[ c_1 e^{jx} + c_2 e^{-jx} \right]$$

or, using the results 1.36 and 1.38,

$$y_t = e^{-x} \left[ c_3 \cos x + c_4 \sin x \right] \quad (1.40)$$

or

$$y_t = e^{-x} A \sin (x + \phi) \quad (1.41)$$

Note, also, that 1.38 could be written in the form

$$y_t = A \cos (x + \theta), \text{ where } \theta = \phi - 90^\circ$$

PROBLEMS: Set I, No. 3, page 1.76; Set II, a only, page 1.85.

### ●1.3.2 PARTICULAR SOLUTION

The particular solution, for our work here, will be obtained by the method of undetermined coefficients. (There are other methods which may be used.) This method consists of assuming a solution of the same general form as the input (forcing function), but with undetermined coefficients. Substitution of this assumed solution into the differential equation then enables us to evaluate these coefficients. The method of undetermined coefficients applies when the forcing function or input is a polynomial, terms of the form  $\sin ax$ ,  $\cos ax$ ,  $e^{ax}$ , or combinations of sums and products of these. The complete solution of the linear differential equation with constant coefficients is then given by 1.18 (that is, the solution to the complementary equation (transient solution), plus the particular solution).

A few remarks are appropriate regarding the second order linear differential equation with constant coefficients. Although the equation is interesting in its own right, it is of particular value to us because it is a mathematical model for several problems of physical interest.

$$a \frac{dy^2}{dx^2} + b \frac{dy}{dx} + cy = F(x) \quad (\text{mathematical model})$$

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + Kx = F(t) \quad (\text{describes a mass spring damper system}) \quad (1.42)$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad (\text{describes a series LRC electrical circuit})$$

Equation 1.42 are all the same mathematically, but are expressed in different notation. Different notations or symbols are employed to emphasize the physical parameters involved, or to force the solution to appear in a form that is easy to interpret. In fact, the similarity of these last two equations may suggest how one might design an electrical circuit to simulate the operation of a mechanical system.

Consider the equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \quad (1.43)$$

We now must solve for the special solution (particular solution) which results from a given input,  $f(x)$ . This particular solution can be found by using various techniques, but we will consider only one, the method of undetermined coefficients. This method consists of assuming a solution form with unspecified constants (undetermined coefficients), and solving for the values of the constants which will satisfy the given differential equation. The method is best described by considering examples.

#### ●1.3.2.1 FORCING FUNCTION = A CONSTANT

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 3y = 6 \quad (1.49)$$

The input is a constant (trivial polynomial), so we assume a solution of form  $y_p = K$ . Obviously,  $d^2K/dx^2 = 0$ , and  $dK/dx = 0$ .

Substituting,

$$0 + 4(0) + 3K = 6$$

$$y_p = K = 2$$

Therefore,  $y_p = 2$  is a particular solution. We note that we can solve the equation

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = 0$$

in operator form

$$(p^2 + 4p + 3)y = 0$$

or

$$p = -1, -3$$

and the transient solution is

$$y_t = c_1 e^{-x} + c_2 e^{-3x}$$

The general solution of 1.44 may be written

$$y = \underbrace{c_1 e^{-x} + c_2 e^{-3x}}_{\text{transient solution}} + \underbrace{2}_{\text{particular (or steady state) solution}}$$

#### ●1.3.2.2 FORCING FUNCTION = A POLYNOMIAL

EXAMPLE

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = x^2 + 2x \quad (1.45)$$

Now the form of  $f(x)$  for 1.45 is a polynomial of second degree, so we assume a particular solution for  $y$  of second degree (that is, let  $y_p = Ax^2 + Bx + C$ ).

Then

$$\frac{dy_p}{dx} = 2Ax + B$$

and

$$\frac{d^2 y_p}{dx^2} = 2A$$

Substituting into 1.45,

$$(2A) + 4(2Ax + B) + 3(Ax^2 + Bx + C) = x^2 + 2x$$

or

$$(3A)x^2 + (8A + 3B)x + (2A + 4B + 3C) = x^2 + 2x$$

Equating like powers of  $x$ ,

$$x^2: 3A = 1$$

$$A = 1/3$$

$$x: 8A + 3B = 2$$

$$3B = 2 - \frac{8}{3}$$

$$B = -2/9$$

$$x^0: 2A + 4B + 3C = 0$$

$$3C = 8/9 - 2/3$$

$$C = 2/27$$

Therefore,

$$y_p = 1/3 x^2 - 2/9 x + 2/27$$

The general solution of 1.45 is given by

$$y = c_1 e^{-x} + c_2 e^{-3x} + 1/3 x^2 - 2/9 x + 2/27$$

since the transient solution is the same as for 1.44. As a general rule, if the forcing function is a polynomial of degree  $n$ , assume a polynomial solution of degree  $n$ .

#### ●1.3.2.3 FORCING FUNCTION = AN EXPONENTIAL

EXAMPLE

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = e^{2x} \quad (1.46)$$

The forcing function is  $e^{2x}$  so we assume a solution of the form

$$y = Ae^{2x}$$

$$\frac{d}{dx} (Ae^{2x}) = 2Ae^{2x}$$

$$\frac{d^2}{dx^2} (Ae^{2x}) = 4Ae^{2x}$$

Substituting in 1.46,

$$4Ae^{2x} + 4(2Ae^{2x}) + 3(Ae^{2x}) = e^{2x}$$

$$e^{2x} (4A + 8A + 3A) = e^{2x}$$

The coefficients on both sides of the equation must be the same. Therefore,  $4A + 8A + 3A = 1$ , or  $15A = 1$ , and  $A = 1/15$ . The particular solution of 1.46 then is  $Y_p = 1/15 e^{2x}$ . The transient solution is still the same as for 1.44. A final example will illustrate a pitfall sometimes encountered using this method.

#### ●1.3.2.4 FORCING FUNCTION - AN EXPONENTIAL (SPECIAL CASE)

EXAMPLE

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = e^{-x} \quad (1.47)$$

The forcing function is  $e^{-x}$ , so we assume a solution of the form  $y = Ae^{-x}$ .

Then

$$\frac{d}{dx} (Ae^{-x}) = -Ae^{-x}$$

and

$$\frac{d^2}{dx^2} (Ae^{-x}) = Ae^{-x}$$

Substituting

$$Ae^{-x} + 4(-Ae^{-x}) + 3(Ae^{-x}) = e^{-x}$$

$$(A - 4A + 3A)e^{-x} = e^{-x}$$

$$(0)e^{-x} = e^{-x}$$

Obviously, this is an incorrect statement. To find where we made our mistake, let's review our procedures.

To solve an equation of the form

$$(p + a)(p + b)y = e^{-ax}$$

we solve the homogeneous equation to get

$$(p + a)(p + b)y = 0$$

$$p = -a, -b$$

$$y_t = c_1 e^{-ax} + c_2 e^{-bx}$$

If we assume  $y_p = Ae^{-ax}$

then

$$\begin{aligned} y = y_t + y_p &= c_1 e^{-ax} + c_2 e^{-bx} + Ae^{-ax} = (c_1 + A)e^{-ax} + c_2 e^{-bx} \\ &= c_3 e^{-ax} + c_2 e^{-bx} \\ &= y_t \end{aligned}$$

However, we have already seen that  $y_t$  is the solution only when the right side of the equation is zero, and will not solve the equation when we have a forcing function. Therefore, we assume a particular solution.

$$y_p = Axe^{-ax}$$

then

$$y = y_p + y_t = c_1 e^{-ax} + c_2 e^{-bx} + Axe^{-ax} = (c_1 + Ax)e^{-ax} + c_2 e^{-bx} \neq y_t$$

Similarly, we could have the equation

$$(p + aj)(p - aj)y = \sin ax$$

with transient solution

$$y_t = c_1 \sin ax + c_2 \cos ax$$

If we assume  $y_p = A \sin ax + B \cos ax$

then

$$y = y_t + y_p = (c_1 + A) \sin ax + (c_2 + B) \cos ax$$

$$+ (c_2 + B) \cos ax$$

$$= c_3 \sin ax + c_4 \cos ax = y_t$$

Therefore, we assume

$$y_p = Ax \sin ax + Bx \cos ax$$

and

$$y = (c_1 + Ax) \sin ax + (c_2 + Bx) \cos ax \neq y_t$$

Note the following, however, with the equation

$$(p + a - jb)(p + a + jb)y = \sin bx$$

$$y_t = e^{-ax} (c_1 \sin bx + c_2 \cos bx)$$

we can assume  $y_p = B \sin bx + C \cos bx$

then

$$y = c_1 e^{-ax} \sin bx + c_2 e^{-ax} \cos bx + B \sin bx + C \cos bx$$

$$y = (c_1 e^{-ax} + B) \sin bx + (c_2 e^{-ax} + C) \cos bx \neq y_t$$

Similarly, if

$$(p + a - jb)(p + a + jb)y = e^{-ax}$$

we could assume

$$y_p = Ae^{-ax}$$

In our example above, equation 1.47, a valid solution can be found by assuming  $y_p = Axe^{-x}$ , then

$$\frac{d}{dx} (Axe^{-x}) = A(-xe^{-x} + e^{-x})$$

and

$$\frac{d^2}{dx^2} (Axe^{-x}) = A(xe^{-x} - 2e^{-x})$$

Substituting

$$A(xe^{-x} - 2e^{-x}) + 4A(-xe^{-x} + e^{-x}) + 3(Axe^{-x}) = e^{-x}$$

$$(A - 4A + 3A)xe^{-x} + (-2A + 4A)e^{-x} = e^{-x}$$

$$(0)xe^{-x} + 2Ae^{-x} = e^{-x}$$

and

$$A = 1/2$$

Thus,

$$y_p = (1/2)xe^{-x}$$

is a particular solution of 1.47, and the general solution is given by:

$$y = c_1e^{-x} + c_2e^{-3x} + 1/2 xe^{-x}$$

The key to successful application of the method of undetermined coefficients is to assume the proper form for a trial particular solution. Table 1 summarizes the results of this discussion.

PROBLEMS: Set II, 1-5, b only, page 1.85.

Table I

Differential equation: $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$	
$f(x)^*$	Assume $y_p^{**}$
1. $\beta$	$A$
2. $\beta x^n$ ( $n$ a positive integer)	$A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1} x + A_n$
3. $\beta e^{rx}$ ( $r$ either real or complex)	$A e^{rx}$
4. $\beta \cos kx$	$A \cos kx + B \sin kx$
5. $\beta \sin kx$	
6. $\beta x^n e^{rx} \cos kx$	$(A_0 x^n + \dots + A_{n-1} x + A_n) e^{rx} \cos kx +$ $+ (B_0 x^n + \dots + B_{n-1} x + B_n) e^{rx} \sin kx$
7. $\beta x^n e^{rx} \sin kx$	

\*When  $f(x)$  consists of a sum of several terms, the appropriate choice for  $y_p$  is the sum of  $y_p$  expressions corresponding to these terms individually.

\*\*Whenever a term in any of the  $y_p$ 's listed in this column duplicates a term already in the complementary function, all terms in that  $y_p$  must be multiplied by the lowest positive integral power of  $x$  sufficient to eliminate the duplication.

### 1.3.3 SOLVING FOR CONSTANTS OF INTEGRATION

As discussed paragraph 1.2, the number of arbitrary constants in the solution of our linear differential equation is equal to the order of the equation. These constants of integration may be determined by initial or boundary conditions. That is, we must know the physical state (position, velocity, etc.) of the system at some time in order to evaluate these constants. Many times these conditions are given at  $t = 0$  (initial conditions), which is frequently called a quiescent system.

It should be emphasized at this point, that the arbitrary constants of the solution are evaluated from the complete solution (transient plus steady state) of the equation.

We shall illustrate this method with an example.

EXAMPLE

$$\ddot{x} + 4\dot{x} + 13x = 3 \quad (1.48)$$

where the dot notation indicates derivatives with respect to time (that is,  $\dot{x} = dx/dt$ ,  $\ddot{x} = d^2x/dt^2$ ). We will assume that the boundary conditions are  $x(0) = 5$ , and  $\dot{x}(0) = 8$ . The transient solution is given by

$$p^2 + 4p + 13 = 0$$

$$p = -2 \pm \sqrt{4 - 13} = -2 \pm j3$$

$$x_t = e^{-2t} (A \cos 3t + B \sin 3t)$$

We assume

$$x_p = D$$

$$\dot{x} = \frac{dx_p}{dt} = 0$$

$$\ddot{x}_p = 0$$

Substituting into 1.48, we get  $D = 3/13$

for a complete solution

$$x(t) = e^{-2t} (A \cos 3t + B \sin 3t) + 3/13$$

To solve for A and B, we will use the initial conditions specified above.

$$x(0) = 5 = A + 3/13$$

or

$$A = 62/13$$

Differentiating the complete solution, we get

$$\dot{x}(t) = e^{-2t} (3B \cos 3t - 3A \sin 3t) - 2e^{-2t} (A \cos 3t + B \sin 3t)$$

Substituting the second initial condition

$$\dot{x}(0) = 8 = 3B - 2A$$

$$B = \frac{76}{13}$$

Therefore, the complete solution to 1.48 with the given initial conditions is

$$x(t) = e^{-2t} [(62/13) \cos 3t + (76/13) \sin 3t] + 3/13$$

We have discussed the first and second order differential equation in some detail. It is of great importance to note that many higher order systems quite naturally decompose into first and second order systems. For example, the study of a third order equation (or system) may be conducted by examining a first and a second order system, a fourth order system analyzed by examining two second order systems, etc. All these cases are handled by solving the characteristic equation to get a transient solution and then obtaining the particular solution by any convenient method.

PROBLEMS: Set II, Nos. 1-5, c only, page 1.85.

## 1.4 APPLICATIONS

Up to this point, we have considered differential equations in general and linear differential equations with constant coefficients in greater detail. We have developed methods for solving first and second order equations of the following type:

$$a \frac{dx}{dt} + bx = f(t) \quad (1.49)$$

$$a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = f(t) \quad (1.50)$$

These two equations are mathematical models or forms. These same forms may be used to describe diverse physical systems. In this section we shall concentrate on the transient response of the systems under investigation, since this area is of primary interest in future studies.

### 1.4.1 FIRST ORDER EQUATION

Consider the following example:

EXAMPLE

$$4\dot{x} + x = 3 \quad (1.51)$$

where

$$\dot{x} = \frac{dx}{dt}$$

Physically, we can let  $x$  represent distance or displacement, and  $t$  represent time. To solve this equation, we find the transient solution by using the homogeneous equation

$$4\dot{x} + x = 0$$

$$(4p + 1)x = 0$$

$$4p + 1 = 0$$

$$p = -1/4$$

Thus

$$x_t = ce^{-t/4}$$

The particular solution is found by assuming

$$x_p = A$$

$$\frac{dx_p}{dt} = 0$$

Substituting

$$A = 3$$

or

$$x_p = 3$$

The complete solution is then

$$x = ce^{-t/4} + 3 \quad (1.52)$$

The first term on the right of 1.52 represents the transient response of the physical system described by equation 1.51, and the second term represents the steady state response if the transient decays. A term useful in describing the physical effect of a negative exponential term is time constant which is denoted by  $\tau$ . We shall define  $\tau$  as

$$\tau \triangleq -\frac{1}{p}$$

Thus, equation 1.52 could be rewritten as

$$x = ce^{-t/\tau} + 3 \quad (1.53)$$

where  $\tau = 4$ .

Note the following points:

1. We only discuss time constants if  $p$  is negative. If  $p$  is positive, the exponent of  $e$  is positive, and the transient solution will not decay.

2. If  $p$  is negative,  $\tau$  is positive.
3.  $\tau$  is the negative reciprocal of  $p$ , so that small numerical values of  $p$  give large numerical values of  $\tau$  (and vice versa).
4. The value of  $\tau$  is the time, in seconds, required for the displacement to decay to  $1/e$  of its original displacement from equilibrium or steady value. To get a better understanding of this statement, let's look at 1.53.

$$x = ce^{-t/\tau} + 3$$

and let  $t = \tau$ . Then

$$x = ce^{-1} + 3 = c \frac{1}{e} + 3$$

Thus, when  $t = \tau$ , the exponential portion of the solution has decayed to  $1/e$  of its original displacement (figure 1.3).

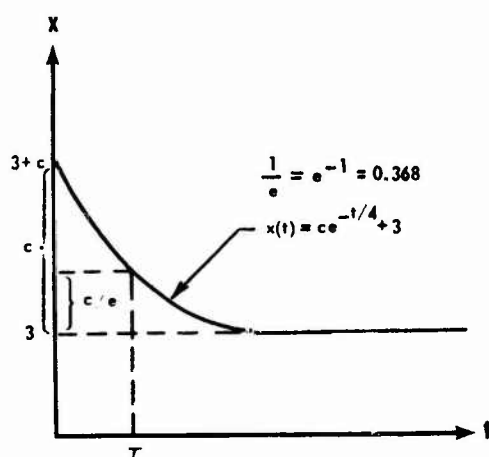


Figure 1.3

Other measures of time are sometimes used to describe the decay of the exponential of a solution. If we let  $T_1$  denote the time it takes for the transient to decay to one-half its original amplitude, then

$$T_1 = 0.693 \tau \quad (1.54)$$

This relationship can be easily shown by investigating

$$x = c_1 e^{-at} + c_2 \quad (1.55)$$

From our definition,  $\tau = 1/a$ . We are looking for  $T_1$ , the value of  $t$  at which  $x_t = 1/2 x_t(0)$ . Solving

$$x_t = c_1 e^{-at}$$

$$1/2 x_t(0) = 1/2 c_1 = c_1 e^{-aT_1}$$

$$e^{-aT_1} = 1/2$$

$$\ln 1/2 = aT_1$$

$$T_1 = \frac{-\ln 1/2}{a} = \frac{.693}{a} = .693\tau$$

Let's complete our solution of 1.51 by specifying a boundary condition and evaluating the arbitrary constant. Let  $x = 0$  at  $t = 0$ .

$$x = ce^{-t/4} + 3$$

$$x(0) = 0 = c + 3$$

$$c = -3$$

Our complete solution for this boundary condition is

$$x = -3e^{-t/4} + 3$$

See figure 1.4.

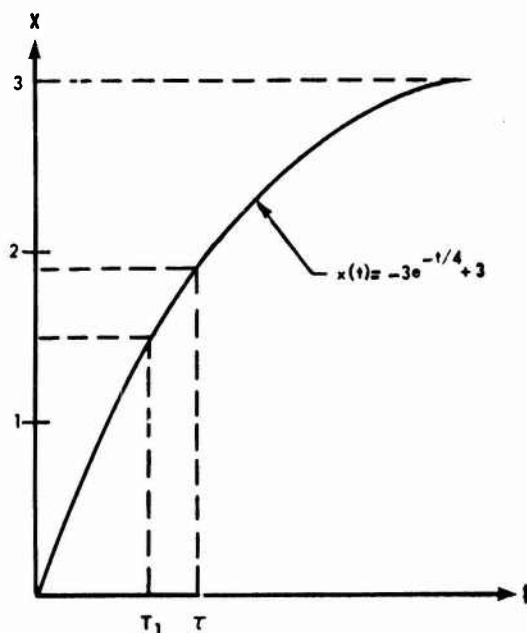


Figure 1.4

## ● 1.4.2 SECOND ORDER EQUATION

Consider an equation of the form 1.50. The characteristic equation (operator equation) can be written:

$$a p^2 + b p + c = 0 \quad (1.56)$$

The roots of this quadratic equation determine the form of the transient solution as we have seen in paragraph 1.3. We will now discuss physical implications of the algebraic property of the roots.

### ● 1.4.2.1 ROOTS REAL AND UNEQUAL

When the roots are real and unequal, the transient solution has the form

$$x_t = c_1 e^{p_1 t} + c_2 e^{p_2 t} \quad (1.57)$$

#### 1.4.2.1.1 Case 1

When  $p_1$  and  $p_2$  are both negative, the system decays and there will be a time constant associated with each exponential (figure 1.5).

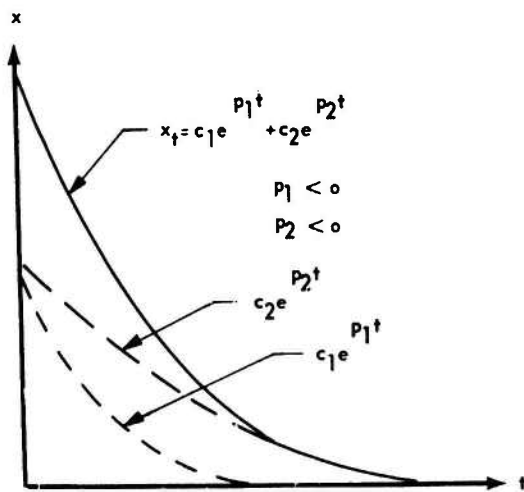


Figure 1.5

#### 1.4.2.1.2 Case 2

When  $p_1$  or  $p_2$  (or both) is positive, the system will generally diverge (figures 1.6 and 1.7).

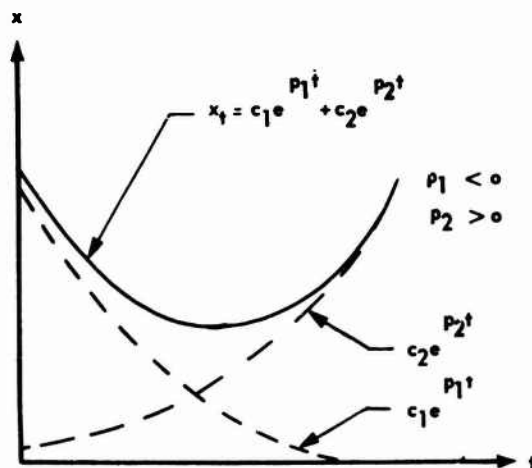


Figure 1.6

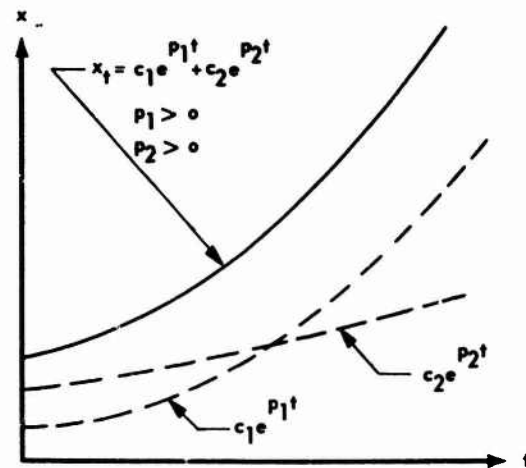


Figure 1.7

#### 1.4.2.1.3 Case 3

Examples where  $p_1$  or  $p_2$  (or both) are zero, are usually not observed in practical cases.

#### ● 1.4.2.2 ROOTS REAL AND EQUAL

When  $p_1 = p_2$ , the transient solution has the form

$$x_t = c_1 e^{pt} + c_2 t e^{pt} \quad (1.58)$$

##### 1.4.2.2.1 Case 1

When  $p$  is negative, the system will usually decay (figure 1.8). (If  $p$  is very small, the system may initially exhibit divergence.)

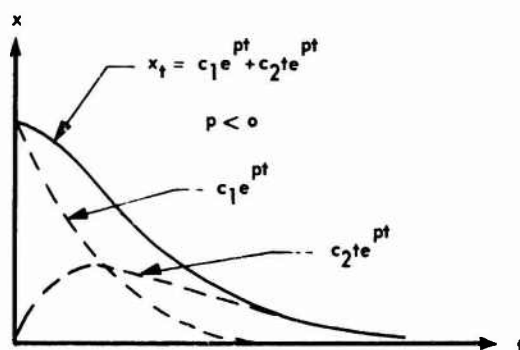


Figure 1.8

#### 1.4.2.2 Case 2

When  $p$  is positive, the system will diverge.

#### ● 1.4.2.3 ROOTS PURELY IMAGINARY

When  $p = \pm jk$ , the transient solution has the form

$$x_t = c_1 \sin kt + c_2 \cos kt \quad (1.59)$$

or

$$x_t = A \sin (kt + \phi) \quad (1.60)$$

or

$$x_t = A \cos (kt + \theta) \quad (1.61)$$

The system executes oscillations of constant amplitude with a frequency  $k$  (figure 1.9).

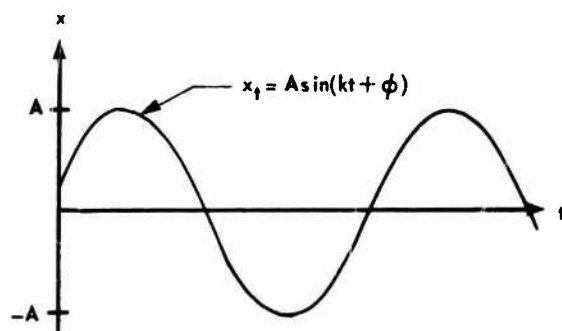


Figure 1.9

#### ● 1.4.2.4 ROOTS COMPLEX CONJUGATES

When the roots are given by  $p = k_1 \pm jk_2$ , the form of the transient solution is

$$x_t = e^{k_1 t} (c_1 \cos k_2 t + c_2 \sin k_2 t) \quad (1.62)$$

or

$$x_t = A e^{k_1 t} \sin (k_2 t + \phi) \quad (1.63)$$

or

$$x_t = A e^{k_1 t} \cos (k_2 t + \theta) \quad (1.64)$$

The system executes periodic oscillations contained in an envelope given by  $x = \pm A e^{k_1 t}$

#### 1.4.2.4.1 Case 1

When  $k_1$  is negative, the system decays (figure 1.10).

#### 1.4.2.4.2 Case 2

When  $k_1$  is positive, the system diverges (figure 1.11).

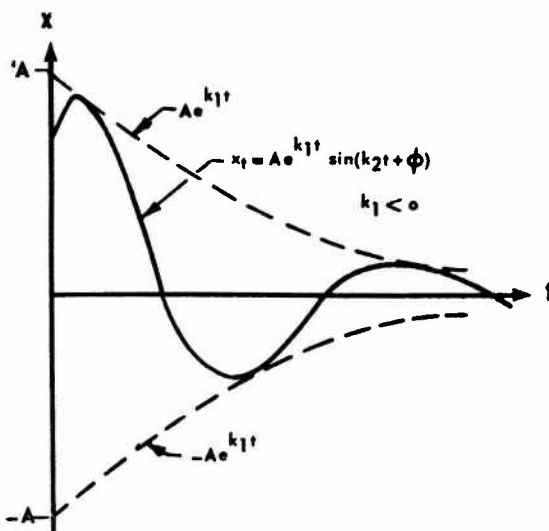


Figure 1.10

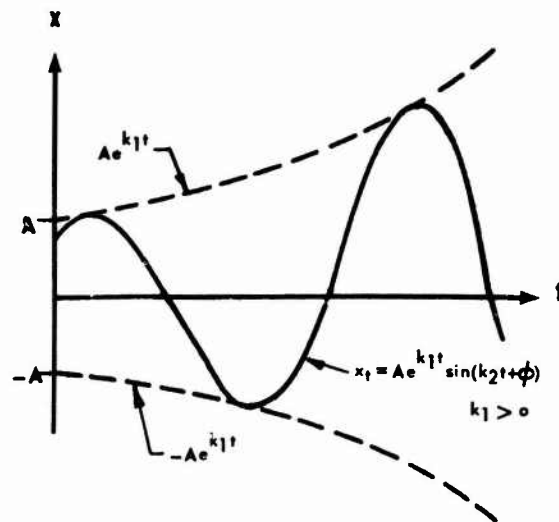


Figure 1.11

The discussion of transient solutions above reveals only part of the picture presented by equation 1.50. We still have the input for forcing function to consider, that is,  $f(t)$ . In practice, a linear system that possesses a divergence (without input) may be changed to a damped system by carefully selecting or controlling the input. Conversely, a nondivergent linear system with weak damping may be made divergent by certain types of inputs.

### ● 1.4.3 SECOND ORDER LINEAR SYSTEMS

Consider the physical model shown in figure 1.12. The system consists of an object suspended by a spring, with a spring constant of  $k$ . The mass may move vertically and is subject to gravity, input, and damping, with the total viscous damping constant equal to  $c$ .

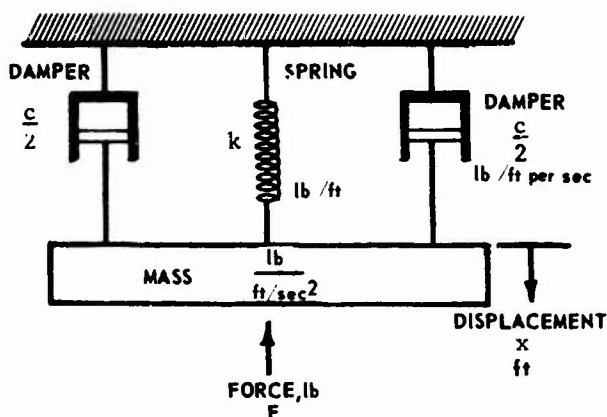


Figure 1.12

The equation for this vibrating system is given by

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad (1.65)$$

The characteristic equation is given by

$$mp^2 + cp + k = 0 \quad (1.66)$$

and the roots of this equation are

$$\begin{aligned} p_{1,2} &= \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ &= \frac{-c}{2m} \pm \sqrt{\frac{k}{m}} \sqrt{\frac{c^2}{4km} - 1} \end{aligned} \quad (1.67)$$

Let us, for simplicity, and for reasons that will be obvious later define three constants

$$\zeta = \frac{c}{2\sqrt{mk}} \quad (1.68)$$

the term  $\zeta$  is called the damping ratio, and is a value which indicates the damping strength in the system.

$$\omega_n = \sqrt{\frac{k}{m}} \quad (1.69)$$

$\omega_n$  is the undamped natural frequency of the system. This is the frequency at which the system would oscillate if there were no damping present.

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (1.70)$$

$\omega_d$  is the damped frequency of the system. It is the frequency at which the system oscillates when a damping ratio of  $\zeta$  is present.

Substituting  $\zeta$  and  $\omega_n$  into 1.67 now gives

$$p_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} \quad (1.71)$$

With these roots, the transient solution becomes

$$\begin{aligned} x_t &= c_1 e^{p_1 t} + c_2 e^{p_2 t} \\ &= e^{-\zeta \omega_n t} [c_1 \cos \omega_n \sqrt{1 - \zeta^2} t + c_2 \sin \omega_n \sqrt{1 - \zeta^2} t] \end{aligned} \quad (1.72)$$

or

$$x_t = A e^{-\zeta \omega_n t} \sin (\omega_n \sqrt{1 - \zeta^2} t + \phi) \quad (1.73)$$

Note that the solution will lie within an exponentially decreasing envelope which has a time constant of  $1/(\zeta \omega_n)$ . This damped oscillation is shown in figure 1.13.

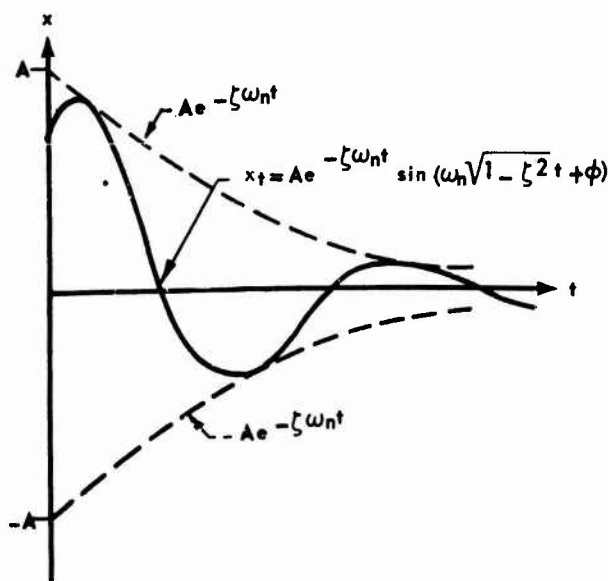


Figure 1.13

If we divide equation 1.65 by  $m$  we obtain

$$x + \frac{c}{m} \dot{x} + \frac{k}{m} x = \frac{f(t)}{m}$$

or, rewriting using  $\omega_n$  and  $\zeta$  defined by 1.68 and 1.69

$$\ddot{x} + 2 \zeta \omega_n \dot{x} + \omega_n^2 x = \frac{f(t)}{m} \quad (1.74)$$

Equation 1.74 is a form of 1.65 that is most useful in analyzing the behavior of any linear system.

A general second order physical system can be compared with mass-spring-damper system. The equation defining the system was

$$m \ddot{x} + c \dot{x} + k x = f(t) \quad (1.65)$$

where we defined the parameters

$$\omega_n = \sqrt{\frac{k}{m}}, \text{ undamped natural frequency}$$

$$\zeta = \frac{c}{2 \sqrt{mk}}, \text{ damping ratio}$$

From equation 1.71 we see that the numerical value of  $\zeta$  is a powerful factor in determining the type of response exhibited by the system.

PROBLEMS: Set II, 1-5, d only, page 1.86.

Let us now consider the physical problem and analyze the various conditions possible. The magnitude and sign of  $\zeta$ , the damping ratio, determine the response properties of the system.

There are five distinct cases which are given names descriptive of the response associated with each case.

1.  $\zeta = 0$ , undamped
2.  $0 < \zeta < 1$ , underdamped
3.  $\zeta = 1$ , critically damped
4.  $\zeta > 1$ , overdamped
5.  $\zeta < 0$ , unstable

We shall now examine each case, making use of equation 1.71

$$p_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} \quad (1.71)$$

● 1.4.3.1 CASE 1:  $\zeta = 0$ , UNDAMPED

For this condition, the roots of the characteristic equation are

$$p_{1,2} = \pm j\omega_n \quad (1.75)$$

giving a transient solution of the form

$$x_t = c_1 \cos \omega_n t + c_2 \sin \omega_n t \quad (1.76)$$

or

$$x_t = A \sin (\omega_n t + \phi) \quad (1.77)$$

showing the system to have the transient response of an undamped sinusoidal oscillation with frequency  $\omega_n$ . (Hence, the designation of  $\omega_n$  as the "undamped natural frequency.") Figure 1.9 shows an undamped system.

Figure 1.14 illustrates typical response for differing values of damping ratios between zero and one.

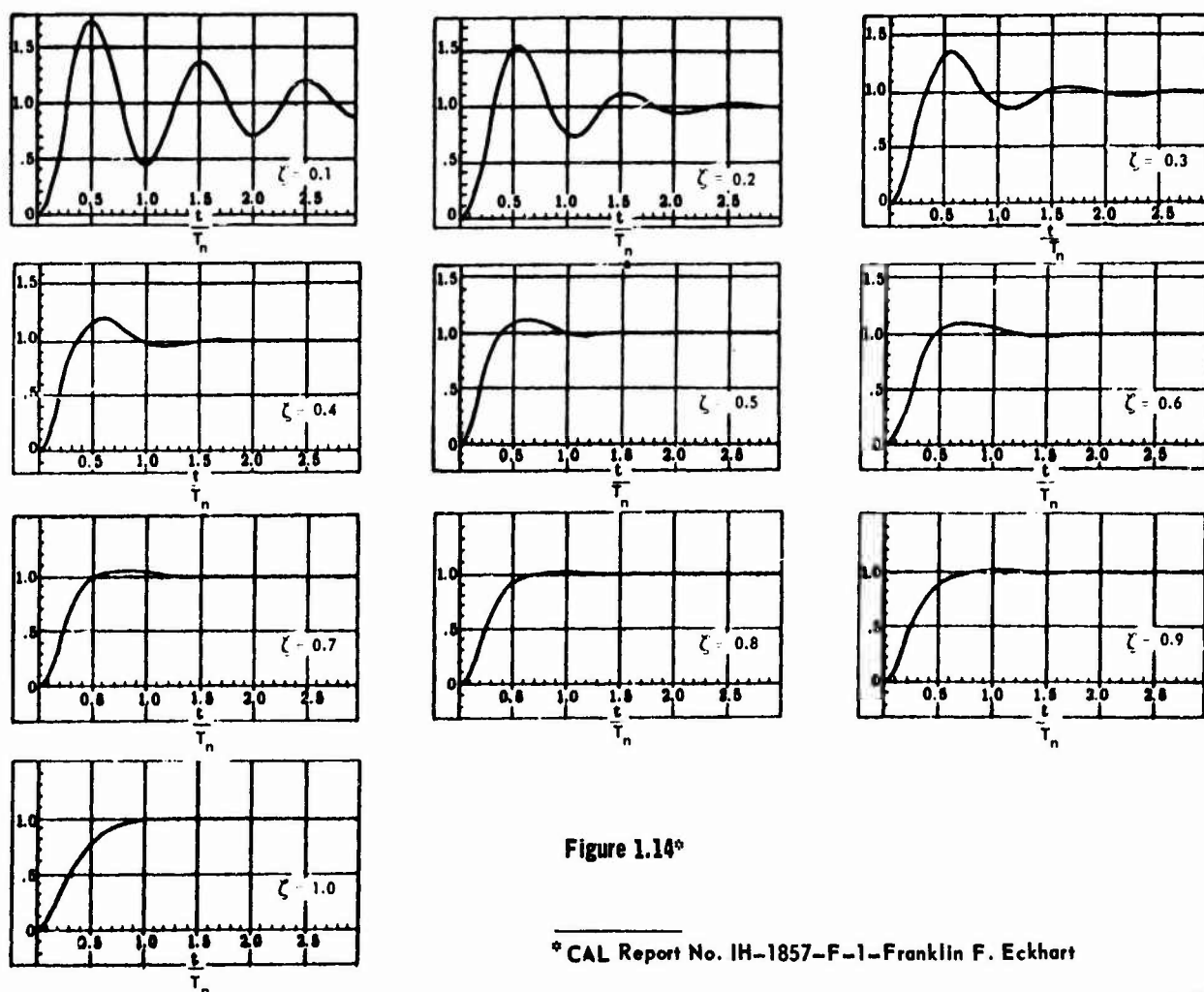


Figure 1.14\*

\* CAL Report No. IH-1857-F-1-Franklin F. Eckhart

● 1.4.3.2 CASE 2:  $0 < \zeta < 1$ , UNDERDAMPED

For this case,  $p$  is given by equation 1.71 and the transient solution has the form

$$x_t = A e^{-\zeta \omega_n t} \sin (\omega_n \sqrt{1 - \zeta^2} t + \phi) \quad (1.78)$$

This solution shows that the system oscillates at the damped frequency,  $\omega_d$ , and is bounded by an exponentially decreasing envelope with time constant  $1/(\zeta \omega_n)$ . Figure 1.14 shows the effect of increasing the damping ratio from 0.1 to 1.0.

● 1.4.3.3 CASE 3:  $\zeta = 1$ , CRITICALLY DAMPED

For this condition, the roots of the characteristic equation are

$$p_{1,2} = -\omega_n \quad (1.79)$$

which gives a transient solution of the form

$$x_t = c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t} \quad (1.80)$$

This is called the critically damped case and generally will not overshoot. It should be noted, however, that large initial values of  $\dot{x}$  can cause one overshoot. Figure 1.14 above shows a response when  $\zeta = 1$ .

● 1.4.3.4 CASE 4:  $\zeta > 1$ , OVERDAMPED

In this case, the characteristic roots are

$$p_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \quad (1.81)$$

which shows that both roots are real and negative. This tells us that the system will have a transient which has an exponential decay without sinusoidal motion. The transient response is given by

$$x_t = c_1 e^{-\omega_n \left[ \zeta - \sqrt{\zeta^2 - 1} \right] t} + c_2 e^{-\omega_n \left[ \zeta + \sqrt{\zeta^2 - 1} \right] t} \quad (1.82)$$

This response can also be written as

$$x_t = c_1 e^{-t/\tau_1} + c_2 e^{-t/\tau_2} \quad (1.83)$$

where  $\tau_1$  and  $\tau_2$  are time constants for each exponential term.

This solution is the sum of two decreasing exponentials, one with time constant  $\tau_1$  and the other with time constant  $\tau_2$ . The smaller the value of  $\tau$ , the quicker the transient decays. Usually the larger the

value of  $\zeta$ , the larger  $\tau_1$  is compared to  $\tau_2$ . For the case  $\zeta > 1$ ,  $\tau_2$  is small in comparison to  $\tau_1$  and can be neglected. The system then behaves like a first order system (that is, the effect of mass can be neglected). This can be seen most readily from equation 1.83. Figure 1.5 shows an overdamped system.

● 1.4.3.5 CASE 5:  $-1 < \zeta < 0$ , UNSTABLE

For this case, the roots of the characteristic equation are

$$p_{1,2} = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \quad (1.84)$$

These roots are the same as for the underdamped case, except that the exponential term in the transient solution shows an exponential increase with time.

$$x_t = e^{-\zeta \omega_n t} [c_1 \cos \omega_n \sqrt{1 - \zeta^2} t + c_2 \sin \omega_n \sqrt{1 - \zeta^2} t] \quad (1.85)$$

Whenever a term appearing in the transient solution grows with time (and especially an exponential growth), the system is generally unstable. This means that whenever the system is disturbed from equilibrium, the disturbance will increase with time. Figure 1.11 shows an unstable system.

● 1.4.3.6 CASE 6:  $\zeta = -1$ , UNSTABLE

For this case, the roots of the characteristic equation are

$$p_{1,2} = +\omega_n$$

and

$$x_t = e^{(\omega_n t)}(c_1 + c_2 t)$$

● 1.4.3.7 CASE 7:  $\zeta < -1$ , UNSTABLE

This case is similar to case 4, except that the system diverges. See figure 1.7.

$$p_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

EXAMPLE

Given

$$\ddot{x} + 4x = 0$$

from equation 1.74

$$\zeta = 0$$

and

$$\omega_n = 2$$

The system is undamped with a solution

$$x_t = A \sin (2t + \phi)$$

where A and  $\phi$  are determined by substituting the boundary conditions into the complete solution.

#### EXAMPLE

Given

$$\ddot{x} + \dot{x} + x = 0$$

from equation 1.73

$$\omega_n = 1$$

and

$$\zeta = 0.5$$

we also know from equation 1.70 that

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.87$$

The system is underdamped with a solution

$$x_t = Ae^{-0.5t} \sin (0.87t + \phi)$$

#### EXAMPLE

Given

$$\frac{\ddot{x}}{4} + \dot{x} + x = 0$$

We multiply 4 to get the equation in the form of equation 1.74.

Then

$$\ddot{x} + 4\dot{x} + 4x = 0$$

and

$$\omega_n = 2$$

$$\zeta = 1$$

The system is critically damped and has a solution given by

$$x_t = c_1 e^{-2t} + c_2 t e^{-2t}$$

EXAMPLE

Given

$$\ddot{x} + 8\dot{x} + 4x = 0$$

we get

$$\omega_n = 2$$

and

$$\zeta = 2$$

The system is overdamped and has a solution

$$x_t = c_1 e^{-7.46t} + c_2 e^{-0.54t}$$

EXAMPLE

Given

$$\ddot{x} - 2\dot{x} + 4x = 0$$

From equation 1.74

$$\omega_n = 2$$

and

$$\zeta = -0.5$$

From equation 1.70

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 1.7$$

The solution is unstable (negative damping) and has the form

$$x_t = A e^t \sin(1.7 t + \phi)$$

● 1.4.38 DAMPING (See figure 1.14)

The best damping ratio for a system is determined by the intended use of the system. If a fast response is desired, and the size and number of overshoots is inconsequential, then we would use a small value of  $\zeta$ . If it is essential that the system not overshoot, and we are not too concerned about response time, we could attempt to use a critically

damped (or even an overdamped) system. The value  $\zeta = 0.7$  is often referred to as the optimum damping ratio since it gives a small overshoot and a relative quick response. It should be noted that "optimum damping ratio" will change as the requirements of the physical system change.

PROBLEMS: Set II, e only, page 1.86.

#### ● 1.4.4 ANALOGOUS SECOND ORDER LINEAR SYSTEMS

##### ● 1.4.4.1 MECHANICAL SYSTEM

The second order equation we have been working with represents the mass-spring-damper system of figure 1.12 and has a differential equation given by

$$m \ddot{x} + c \dot{x} + k x = f(t) \quad (1.86)$$

where

$m$  = mass

$c$  = damping coefficient

$k$  = spring constant

and we defined

$$\omega_n = \sqrt{\frac{k}{m}} \quad (1.87)$$

$$\zeta = \frac{c}{2\sqrt{mk}} \quad (1.88)$$

and thus

$$2\zeta\omega_n = \frac{c}{m}$$

Equation 1.86 may then be rewritten,

$$\ddot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = f_1(t) \quad (1.89)$$

where

$$f_1(t) = \frac{f(t)}{m}$$

or

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_1(t) \quad (1.90)$$

#### ● 1.4.4.2 ELECTRICAL SYSTEM

The second order equation can also be applied to the series LRC circuit shown in figure 1.15.

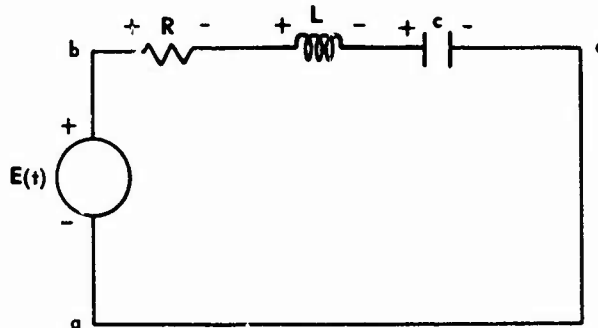


Figure 1.15

where

$L$  = inductance

$R$  = resistance

$C$  = capacitance

$q$  = charge

$i$  = current

Assume  $q(0) = \dot{q}(0) = 0$ , then Kirchhoff's voltage law gives

$$\sum V_{abd} = 0$$

or

$$E(t) - V_R - V_L - V_C = 0$$

$$E(t) - iR - L \frac{di}{dt} - \frac{1}{C} \int_0^t i dt = 0$$

Since

$$i = \frac{dq}{dt}$$

$$E(t) = Lq + R\dot{q} + \frac{q}{C} \quad (1.91)$$

We now define

$$\omega_n = \sqrt{\frac{1}{LC}}$$

$$\zeta = \frac{R}{2\sqrt{L/C}}$$

$$2\zeta\omega_n = \frac{R}{L}$$

Using these parameters, equation 1.91 can be written

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = E_1(t) = \frac{E(t)}{L} \quad (1.92)$$

#### ● 1.4.4.3 SERVOMECHANISMS

For control systems work, the second order equation is

$$I\ddot{\theta}_o + f\dot{\theta}_o + u\theta_o = u\theta_i \quad (1.93)$$

where

$I$  = inertia

$f$  = friction

$u$  = gain

$\theta_i$  = input

$\theta_o$  = output

Rearranging 1.93 we have

$$\ddot{\theta}_o + \frac{f}{I}\dot{\theta}_o + \frac{u}{I}\theta_o = \frac{u}{I}\theta_i \quad (1.94)$$

or

$$\ddot{\theta}_o + 2\zeta\omega_n\dot{\theta}_o + \omega_n^2\theta_o = \omega_n^2\theta_i \quad (1.95)$$

where we define

$$\omega_n = \sqrt{\frac{u}{I}}$$

$$\zeta = \frac{f}{2\sqrt{uI}}$$

Thus, we see that we can generally write any second order differential equation in the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t) \quad (1.96)$$

where each term has the same qualitative significance, but different physical significance.

## ■ 1.5 LAPLACE TRANSFORMS

We have developed a technique for solving linear differential equations with constant coefficients, with and without inputs or forcing functions. We have admitted that our method has limitations. It is suited for differential equations with inputs of only certain forms. Further, the solution procedure requires that the student stay constantly alert for special cases that require careful handling. We accepted these "bookkeeping" chores because our solution procedures had the remarkable property of changing or "transforming" a problem of integration into a problem in algebra. (That is, solving a quadratic equation in the case of second order differential equations.) This was accomplished by making an assumption involving the number  $e$ , as follows:

Given

$$a\ddot{x} + b\dot{x} + cx = 0 \quad (1.97)$$

Assume

$$x = e^{mt} \quad (1.98)$$

Substituting

$$am^2 e^{mt} + bme^{mt} + ce^{mt} = 0 \quad (1.99)$$

and

$$e^{mt} (am^2 + bm + c) = 0 \quad (1.100)$$

led us to assert that 1.98 would produce a solution if  $m$  were a root of the characteristic equation

$$am^2 + bm + c = 0 \quad (1.101)$$

We then introduced an operator,  $p = d/dt$ , and noted a short cut (book-keeping coincidence) to writing the characteristic equation 1.101 as

$$ap^2 + bp + c = 0 \quad (1.102)$$

which we then solved for  $p$  to give solution of the form

$$x = c_1 e^{p_1 t} + c_2 e^{p_2 t} \quad (1.103)$$

Of course, the great shortcoming of this method was that it did not provide a solution to an equation of the form

$$a\ddot{x} + b\dot{x} + cx = f(t) \quad (1.104)$$

It only worked for the homogeneous equation. Still, we were able to patch together a solution by obtaining a particular solution (using still another technique) and adding it to the "transient" solution of the homogeneous equation. It should be appreciated that the method of undetermined coefficients also provided a solution by algebraic manipulation.

Suppose we were adventurous enough to inquire further. We ask, "Does there exist a technique which would exchange (transform) the whole differential equation, including the input, into an algebra problem?" The answer is a qualified "Yes." Fortunately, the "Yes" answer applies to the types of equations with which we have been working.

In equation 1.104,  $x$  is a function of  $t$ . To emphasize this, we rewrite 1.104 as:

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = f(t) \quad (1.105)$$

Suppose we multiply each term of 1.105 by  $e^{mt}$ , giving us:

$$a\ddot{x}(t)e^{mt} + b\dot{x}(t)e^{mt} + cx(t)e^{mt} = f(t)e^{mt} \quad (1.106)$$

Now, a most remarkable feature begins to emerge. It so happens that 1.106 can be integrated term by term on both sides of the equation to produce an algebraic expression in  $m$ . The algebraic expression can then be manipulated to obtain eventually the solution of 1.106.

The preceding statements have omitted many details, but express the method of solution we now seek to develop. Our new "fudge factor",  $e^{mt}$ , should be distinguished from the previous technique for solving the homogeneous equation, so we shall replace the  $m$  by the term,  $-s$ . The reason for the minus sign will become apparent later. If we are to integrate the terms of 1.106, we shall need limits of integration. In most physical problems we are interested in events that take place subsequent to a given starting time which we shall call  $t = 0$ . Since we are unsure of the duration of significant events, we shall sum up the composite of effects from time  $t = 0$  to time  $t = \infty$  (that should cover the field). So now equation 1.106 becomes

$$\begin{aligned} \int_0^{\infty} a\ddot{x}(t) e^{-st} dt + \int_0^{\infty} b \dot{x}(t) e^{-st} dt + \int_0^{\infty} c x(t) e^{-st} dt \\ = \int_0^{\infty} f(t) e^{-st} dt \end{aligned} \quad (1.107)$$

Equation 1.107 is called the Laplace transform of equation 1.105.

There is one small problem. How do we integrate these terms? We now focus our attention upon this problem.

### 1.5.1 FINDING THE LAPLACE TRANSFORM OF A DIFFERENTIAL EQUATION

We now attempt to find the integrals of the terms of the differential equation 1.107. The big unanswered question posed by equation 1.107 is "What is  $x(t)$ ?" (that is,  $x(t)$  is an unknown). Thus,

$$\int_0^{\infty} x(t) e^{-st} dt = L\{x(t)\} = X(s) \quad (1.108)$$

$X(s)$  must, for the present, remain an unknown. (Remember that  $m$  was carried along as an unknown until the characteristic equation evolved, at which time we solved for  $m$  explicitly.) Since 1.108 transforms  $x(t)$  into a function of the variable,  $s$ , we shall say

$$\int_0^{\infty} c x(t) e^{-st} dt = c \int_0^{\infty} x(t) e^{-st} dt = cX(s) \quad (1.109)$$

and be content to carry along  $X(s)$  until such time that we can solve for it.

Now consider the second term,  $b \dot{x}(t)$ . We want to find:

$$\int_0^{\infty} b \dot{x}(t) e^{-st} dt = b \int_0^{\infty} \dot{x}(t) e^{-st} dt \quad (1.110)$$

To solve 1.110 we call upon a useful tool known as integration by parts.

Recall

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du \quad (1.111)$$

Applying this tool to equation 1.110 we let

$$u = e^{-st}$$

and

$$dv = \dot{x}(t) dt$$

then

$$du = -se^{-st} dt$$

and

$$v = x(t)$$

Putting these values into 1.111 and integrating from  $t = 0$  to  $t = \infty$ ,

$$\begin{aligned}\int_0^{\infty} \dot{x}(t)e^{-st} dt &= \left[ x(t)e^{-st} \right]_0^{\infty} - \int_0^{\infty} x(t) \left[ -se^{-st} \right] dt \\ &= \left[ x(t)e^{-st} \right]_0^{\infty} + s \int_0^{\infty} x(t)e^{-st} dt \\ &= \left[ x(t)e^{-st} \right]_0^{\infty} + sX(s)\end{aligned}\quad (1.112)$$

Now

$$\left[ x(t)e^{-st} \right]_0^{\infty} = \lim_{t \rightarrow \infty} x(t)e^{-st} - x(0) \quad (1.113)$$

and we shall assume that the term  $e^{-st}$  "dominates" the term  $x(t)$  as  $t \rightarrow \infty$ .

Thus,  $\lim_{t \rightarrow \infty} x(t)e^{-st} = 0$ , and equation 1.111 becomes

$$\int_0^{\infty} \dot{x}(t)e^{-st} dt = 0 - x(0) + sX(s) = sX(s) - x(0) \quad (1.114)$$

Equations 1.109 and 1.114 can be abbreviated by using the letter  $L$  to signify Laplace transformations.

$$L\{x(t)\} = X(s)$$

$$L\{cx(t)\} = cX(s) \quad (1.115)$$

$$L\{\dot{x}(t)\} = sX(s) - x(0)$$

$$L\{b\dot{x}(t)\} = b[sX(s) - x(0)] \quad (1.116)$$

Equation 1.116 can be extended to higher order derivatives. Such an extension gives

$$L\{a\ddot{x}(t)\} = a[s^2X(s) - sx(0) - \dot{x}(0)] \quad (1.117)$$

Returning to equation 1.107, we note that we have found the Laplace transforms of all the terms except the forcing function. To solve this transform, the forcing function must be specified. We shall consider a few typical functions and illustrate, by example, the technique for finding the Laplace transform.

#### EXAMPLE

$$f(t) = A = \text{constant}$$

Then

$$L\{A\} = \int_0^{\infty} Ae^{-st} dt = \frac{A}{-s} \int_0^{\infty} e^{-st} (-s dt) = -\frac{A}{s} e^{-st} \Big|_0^{\infty}$$

or

$$L\{A\} = \frac{A}{s} \quad (1.118)$$

EXAMPLE

$$f(t) = t$$

Then

$$L\{t\} = \int_0^{\infty} te^{-st} dt$$

To integrate by parts, we let

$$u = t$$

$$dv = e^{-st} dt$$

Then

$$du = dt$$

$$v = -\frac{1}{s} e^{-st}$$

Substituting into 1.110

$$\begin{aligned} \int_0^{\infty} te^{-st} dt &= \left[ \frac{-te}{s} \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \\ &= 0 - \left[ \frac{1}{s^2} e^{-st} \right]_0^{\infty} = 0 + \frac{1}{s^2} \end{aligned}$$

or

$$L\{t\} = \frac{1}{s^2} \quad (1.119)$$

EXAMPLE

$$f(t) = e^{2t}$$

Then

$$L\{e^{2t}\} = \int_0^{\infty} e^{2t} e^{-st} dt = \int_0^{\infty} e^{(2-s)t} dt = \frac{1}{s-2}$$

or

$$L\{e^{2t}\} = \frac{1}{s-2} \quad (1.120)$$

EXAMPLE

$$f(t) = \sin at$$

Then

$$L\{\sin at\} = \int_0^{\infty} \sin at e^{-st} dt$$

Integrate by parts, letting

$$u = \sin at$$

$$dv = e^{-st} dt$$

Then

$$du = a \cos at dt$$

$$v = -\frac{1}{s} e^{-st}$$

Substituting into 1.111

$$\int_0^{\infty} \sin t e^{-st} = \left. \frac{-(\sin at)(e^{-st})}{s} \right]_0^{\infty} + \frac{a}{s} \int_0^{\infty} \cos at e^{-st} dt$$

or

$$\int_0^{\infty} \sin at e^{-st} dt = 0 + \frac{a}{s} \int_0^{\infty} \cos at e^{-st} dt \quad (1.121)$$

The expression  $\cos at e^{-st} dt$  can also be integrated by parts, letting

$$u = \cos at$$

$$dv = e^{-st} dt$$

and

$$du = -a \sin at dt$$

$$v = \frac{1}{s} e^{-st}$$

Giving

$$\int_0^{\infty} \cos at e^{-st} dt = \left. \frac{-(\cos at)(e^{-st})}{s} \right]_0^{\infty} - \frac{a}{s} \int_0^{\infty} \sin at e^{-st} dt$$

or

$$\int_0^{\infty} \cos at e^{-st} dt = \frac{1}{s} - \frac{a}{s} L(\sin t) \quad (1.122)$$

Substituting 1.122 into 1.121 gives

$$L(\sin at) = 0 + \frac{a}{s} \left[ \frac{1}{s} - \frac{a}{s} L(\sin at) \right] = \frac{a}{s^2} - \frac{a^2}{s^2} L(\sin at)$$

which "obviously" yields

$$L(\sin at) = \frac{a}{s^2 + a^2} \quad (1.123)$$

Also note that 1.121 may be written as

$$L(\sin at) = \frac{a}{s} L(\cos at)$$

which yields

$$L(\cos at) = \frac{s}{s^2 + a^2}$$

The Laplace transforms of more complicated functions may be quite tedious to derive, but the procedure is similar to that above. Fortunately, it is not necessary to derive Laplace transforms each time we use them. Extensive tables of transforms exist in most advanced mathematics and control system textbooks.

We originally asserted that the Laplace transform was going to assist in the solution of a differential equation. The technique is best described by an example.

#### EXAMPLE

Given

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t}$$

with conditions  $x(0) = 1$ ,  $\dot{x}(0) = -4$ . Taking the Laplace transform of the equation gives

$$s^2 X(s) - sx(0) - \dot{x}(0) + 4[sX(s) - x(0)] + 4X(s) = \frac{4}{s-2}$$

or

$$[s^2 + 4s + 4] X(s) + [-s + 4 - 4] = \frac{4}{s-2}$$

Solving for  $X(s)$ ,

$$X(s) = \frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} \quad (1.124)$$

In order to continue with our solution, it is necessary that we discuss partial fraction expansions.

PROBLEMS: Set III, page 1.100.

### ● 1.5.2 PARTIAL FRACTIONS

The method of partial fractions enables us to separate a complicated rational fraction into a sum of simpler fractions. Suppose we are given a fraction of two polynomials in a variable,  $s$ . Suppose the fraction is proper (the degree of the numerator is less than the degree of the denominator). If it is not proper, we make it proper by dividing the fraction and then consider the remainder expression. There occur several cases:

#### ● 1.5.2.1 CASE 1: DISTINCT LINEAR FACTORS

To each linear factor such as  $(as + b)$ , occurring once in the denominator, there corresponds a single partial fraction of the form,  $A/(as + b)$ , where  $A$  is a constant to be determined.

EXAMPLE

$$\frac{7s - 4}{s(s - 1)(s + 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 2} \quad (1.125)$$

#### ● 1.5.2.2 CASE 2: REPEATED LINEAR FACTORS

To each linear factor,  $(as + b)$ , occurring  $n$  times in the denominator there corresponds a set of  $n$  partial fractions.

EXAMPLE

$$\frac{s^2 - 9s + 17}{(s - 2)^2(s + 1)} = \frac{A}{(s + 1)} + \frac{B}{(s - 2)} + \frac{C}{(s - 2)^2} \quad (1.126)$$

(where  $A$ ,  $B$ , and  $C$  are constants to be determined)

#### ● 1.5.2.3 CASE 3: DISTINCT QUADRATIC FACTORS

To each irreducible quadratic factor,  $as^2 + bs + c$ , occurring once in the denominator, there corresponds a single partial fraction of the form,  $(As + B)/(as^2 + bs + c)$ , where  $A$  and  $B$  are constants to be determined.

# EXAMPLE

$$\frac{3s^2 + 5s + 8}{(s + 2)(s^2 + 1)} + \frac{A}{s + 2} + \frac{Bs + C}{s^2 + 1} \quad (1.127)$$

## ● 1.5.2.4 CASE 4: REPEATED QUADRATIC FACTORS

To each irreducible quadratic factor,  $as^2 + bs + c$ , occurring  $n$  times in the denominator, there corresponds a set of  $n$  partial fractions.

# EXAMPLE

$$\frac{10s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A}{s - 4} + \frac{Bs + C}{s^2 + 4} + \frac{Ds + E}{(s^2 + 4)^2} \quad (1.128)$$

(where  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  are constants to be determined)

The "brute-force" technique for finding the constants will be illustrated by solving 1.128. Start by finding the common denominator on the right side of 1.128.

$$\frac{10s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A(s^2 + 4)^2 + (Bs + C)(s - 4)(s^2 + 4) + (Ds + E)(s - 4)}{(s - 4)(s^2 + 4)^2} \quad (1.129)$$

Then set the numerators equal to each other

$$10s^2 + s + 36 = A(s^2 + 4)^2 + (Bs + C)(s^2 + 4)(s - 4) + (Ds + E)(s - 4) \quad (1.130)$$

and, without justifying the statement, we shall assert that 1.130 must hold for all values of  $s$ . Now substitute enough values of  $s$  into 1.130 to find the constants.

1. Suppose  $s = 4$ , then 1.130 becomes

$$(10)(16) + 4 + 36 = 400A$$

and

$$A = 1/2$$

2. Suppose  $s = 2j$ , then 1.130 becomes

$$-40 + 2j + 36 = -4D + 2jE - 8jD - 4E$$

$$-4 + 2j = -4(D + E) + 2j(E - 4D)$$

The real and imaginary parts must be equal to their counterparts on the opposite side of the equal sign, thus

$$(D + E) = 1$$

and

$$E - 4D = 1$$

or

$$D = 0$$

and

$$E = 1$$

3. Now let  $s = 0$ , then 1.130 becomes

$$36 = 16A - 16(C) - 4E$$

and from above

$$A = 1/2, E = 1$$

hence

$$36 = 8 - 16C - 4$$

and

$$C = -2$$

4. Let  $s = 1$ , then 1.130 becomes

$$47 = 25(1/2) + (B - 2)(-15) - 3$$

$$94 = 25 - 30B + 60 - 6,$$

or

$$\text{Th } B = -1/2$$

Then 1.129 may be written

$$\frac{10s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = 1/2 \left( \frac{1}{s - 4} \right) - 1/2 \left( \frac{s + 4}{s^2 + 4} \right) + \frac{1}{(s^2 + 4)^2}$$

Let's continue with our attempt to solve the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t}$$

We have transformed the equation (and substituted initial conditions) to get

$$X(s) = \frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} \quad (1.124)$$

We now expand by partial fractions

$$\frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} = \frac{A}{s - 2} + \frac{B}{s + 2} + \frac{C}{(s + 2)^2} \quad (1.132)$$

Taking the common demoninator, and setting numerators equal

$$s^2 - 2s + 4 = A(s + 2)^2 + B(s + 2)(s - 2) + C(s - 2) \quad (1.133)$$

We can now substitute different values of  $s$  into this equation and solve for the constants. An alternate method of solving for these constants exists, however, and we will demonstrate this new approach. If we multiply out the right side of 1.133 we get

$$\begin{aligned} s^2 - 2s + 4 &= As^2 + 4As + 4A + Bs^2 - 4B + Cs - 2C \\ &= (A + B)s^2 + (4A + C)s + (4A - 4B - 2C) \end{aligned}$$

Now the coefficients of like powers of  $s$  on both sides of the equation must be equal (that is, the coefficient of  $s^2$  on the left side equals the coefficient of  $s^2$  on the right side, etc.) Equating gives

$$s^2 : 1 = A + B$$

$$s^1 : -2 = 4A + C$$

$$s^0 : 4 = 4A - 4B - 2C$$

Solving, we get

$$A = 1/4$$

$$B = 3/4$$

$$C = -3$$

Substituting into 1.132, we get

$$X(s) = 1/4 \left( \frac{1}{s - 2} \right) + 3/4 \left( \frac{1}{s + 2} \right) - 3 \left( \frac{1}{(s + 2)^2} \right) \quad (1.134)$$

### ● 1.5.3 HEAVISIDE EXPANSION THEOREMS FOR ANY $F(s)$

#### ● 1.5.31 CASE 1: DISTINCT LINEAR FACTORS

If the denominator  $F(s)$  has a distinct linear factor,  $(s - a)$ , we find the constant for that factor by multiplying  $F(s)$  by  $(s - a)$ , and then evaluate the remainder of  $F(s)$  at  $s = a$ .

$$F(s) = \frac{A}{s-a} + \dots$$

$$A = (s-a) F(s) \Big|_{s=a}$$

#### EXAMPLE

$$F(s) = \frac{7s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$A = sF(s) \Big|_{s=0} = \frac{7s-4}{(s-1)(s+2)} \Big|_{s=0} = \frac{4}{(-1)(2)} = 2$$

$$B = (s-1)F(s) \Big|_{s=1} = \frac{7s-4}{s(s+2)} \Big|_{s=1} = \frac{7-4}{(1)(3)} = 1$$

$$C = (s+2)F(s) \Big|_{s=-2} = \frac{7s-4}{s(s-1)} \Big|_{s=-2} = \frac{-14-4}{(-2)(-3)} = -3$$

See case 4 also.

#### ● 1.5.3.2 CASE 2: REPEATED LINEAR FACTORS

If the denominator of  $F(s)$  has any repeated linear factors, they must be treated in a special manner.

$$F(s) = \frac{A}{(s-a)^1} + \frac{B}{(s-a)^{n-1}} + \frac{C}{(s-a)^{n-2}} + \dots + \frac{Z}{(s-a)}$$

$$\text{Let } \phi(s) = (s-a)^n F(s)$$

Then

$$A = \phi(s) \Big|_{s=a}$$

$$B = d\phi(s)/ds \Big|_{s=a}$$

$$C = \frac{1}{2!} d^2\phi(s)/ds^2 \Big|_{s=a}$$

$$Z^* = \frac{1}{k!} \frac{d^k\phi(s)}{ds^k} \Big|_{s=a}$$

\*Note: This formula is good for all constants except A above.

where  $k = 1, 2, \dots, n - 1$

where the derivatives of  $\phi(s)$  are obtained by using

$$\frac{d}{ds} \left( \frac{u}{v} \right) = \frac{vdu - u dv}{v^2}$$

For example,

$$F(s) = \frac{s^2 - 9s + 17}{(s - 2)^2 (s + 1)} = \frac{A}{s + 1} + \frac{B}{(s - 2)^2} + \frac{C}{s - 2}$$

$$A = (s + 1) F(s) \Big|_{s = -1} = \frac{s^2 - 9s + 17}{(s - 2)^2} \Big|_{s = -1} = \frac{1 + 9 + 17}{(-3)^2} = \frac{27}{9}$$

$$B = \phi(s) \Big|_{s = 2} = \frac{s^2 - 9s + 17}{s + 1} \Big|_{s = 2} = \frac{4 - 18 + 17}{3} = 1$$

$$C = \frac{d\phi(s)}{ds} \Big|_{s = 2} = \frac{(s + 1)(2s - 9) - (s^2 - 9s + 17)(1)}{(s + 1)^2} \Big|_{s = 2} \\ = \frac{(3)(-5) - (4 - 18 + 17)}{(3)^2} = \frac{-18}{9} = -2$$

See case 4 also.

#### ● 1.5.3.3 CASE 3: DISTINCT QUADRATIC FACTORS

If the denominator of  $F(s)$  contains a distinct quadratic factor  $(s + a)^2 + b^2$ , we will again multiply  $F(s)$  by  $(s + a)^2 + b^2$ , and evaluate the remainder of  $F(s)$ ,  $\phi(s)$ , at  $s = -a + jb$  and use real and imaginary parts of  $\phi(s)$  to obtain the two constants.

$$F(s) = \frac{As + B}{(s + a)^2 + b^2}$$

Let

$$\phi(s) = \left[ (s + a)^2 + b^2 \right] F(s)$$

and compute

$$\phi_r + j\phi_i = \phi(s) \Big|_{s = -a + jb}$$

Then

$$A = \frac{\phi_i}{b} \quad B = \frac{b\phi_r + a\phi_i}{b}$$

For example,

$$F(s) = \frac{4s^2 + 19s + 32}{(s+2) \left[ (s+3)^2 + 4 \right]} = \frac{As + B}{(s+3)^2 + 4} + \frac{C}{s+2}$$

$$\left. \begin{aligned} \phi_r + j\phi_i = \phi(s) \end{aligned} \right]_{s = -3 + j2} = \left. \frac{4s^2 + 19s + 32}{s+2} \right]_{s = -3 + j2}$$

$$= \frac{-5 - j10}{-1 + j2} \cdot \frac{-1 - j2}{-1 - j2} = -3 + j4$$

$$\phi_r = -3 \quad \phi_i = 4 \quad a = 3 \quad b = 2$$

$$A = \frac{4}{2} = 2 \quad B = \frac{-6 + 12}{2} = 3$$

See case 4 also.

#### ●1.5.3.4 CASE 4: REPEATED QUADRATIC FACTORS (AND ANY OTHER CASE)

Procedures similar to those used in the previous cases exist for this case, but they are too cumbersome for most applications. The following procedures will work for any combination of linear and quadratic factors:

$$F(s) = \frac{10s^2 + s + 36}{(s-4)(s^2+4)^2} = \frac{A}{s-4} + \frac{Bs+C}{s^2+4} + \frac{Ds+E}{(s^2+4)^2}$$

Put the right-hand side of the equation over a common denominator and then set the two numerators equal.

$$10s^2 + s + 36 = A(s^2 + 4)^2 + (Bs + C)(s - 4)(s^2 + 4) + (Ds + E)(s - 4) \quad (1.135)$$

which will be a true equation for all values of  $s$ .

#### ●1.5.3.5 PROCEDURES

The following steps may be done in any order, and in combination with the procedures in cases 1 through 3.

1. Since equation 1.135 is true for all values of  $s$ , choosing specific values of  $s$ , five in this case, and substituting into equation 1.135 will give you five equations in five unknowns, which can be solved simultaneously for the constants.
2. Expand the right hand side of equation 1.135 and find the coefficients of each power of  $s$ . These coefficients must be the same on both sides of the equation (that is,  $s^4$ :  $B = 0$ ;  $s^0$ :  $4A - 16C - 4E = 36$ ).
3. Let  $s$  equal an imaginary or complex number and substitute into equation 1.135. The real parts on both sides of the equation must be equal, and so must the imaginary parts.

Examining equation 1.135, we will use a combination of procedures. First, find  $A$  by using case 1

$$A = (s - 4) F(s) \Big|_{s=4} = \frac{10(16) + 4 + 36}{(16 + 4)^2} = \frac{200}{400} = \frac{1}{2}$$

If we let  $s = j2$ , the only non-zero term will be the one with  $D$  and  $E$ . Letting  $s$  be a complex number will give us two equations for  $D$  and  $E$ , which we can solve simultaneously.

$$10(j2)^2 + j2 + 36 = (j2D + E)(j2 - 4)$$

$$-4 + j2 = -4D - 4E - j8D + j2E$$

Now set real and imaginary parts equal.

$$-4D - 4E = -4$$

$$-8D + 2E = 2$$

$$-2D - 2E = -2$$

$$-8D + 2E = 2$$

Adding the two equations together gives:

$$-10D = 0$$

$$\underline{D = 0}$$

Then

$$2E = 2$$

$$\underline{E = 1}$$

To find  $C$ , let  $s = 0$  in equation 1.135

$$36 = 16A - 16C - 4E = 16(1/2) - 16C - 4(1)$$

$$16C = 8 - 4 - 36 = -32$$

$$\underline{C = -2}$$

To find B, let  $s = 1$

$$10 + 1 + 36 = 25A + (B + C)(-3)(5) + (D + E)(-3)$$

$$47 = 25(1/2) - 15B - 15(-2) - 3$$

$$15B = 12 \frac{1}{2} + 30 - 3 - 47 = 42 \frac{1}{2} - 50 = -7 \frac{1}{2}$$

$$\underline{B = -1/2}$$

An alternate way to find B is to calculate the coefficient of  $s^4$  on the right-hand side of equation 1.135, and then set it equal to the coefficient of  $s^4$  on the left hand side of the equation. Then we get

$$A + B = 0$$

$$\underline{B = -A = -1/2}$$

To complete our solution, we must convert (transform) back into the time domain. The operation which converts a function  $X(s)$  back to a function of time is called the inverse Laplace transformation.

$$L^{-1} \left\{ L\{x(t)\} \right\} = L^{-1} \{X(s)\} = x(t) \quad (1.136)$$

The inverse Laplace transformation can be solved directly

$$x(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} X(s) e^{st} ds \quad (1.137)$$

(where C is a real constant)

This integral, 1.137, is hardly ever used because the Laplace transform is unique and, therefore, generally  $X(s)$  can be recognized as the Laplace transform of some known  $x(t)$ . In practice, tables of transform pairs (as found in most mathematics texts) will suffice to find the inverse of  $X(s)$  (see table II, page 1.106, for some transform pairs).

Using a suitable transform table, the inverse of 1.134 can easily be found to give us a solution

$$x(t) = 1/4 e^{2t} + 3/4 e^{-2t} - 3te^{-2t} \quad (1.138)$$

PROBLEMS: Set IV, A, page 1.106.

#### ● 1.5.4 PROPERTIES OF LAPLACE TRANSFORMS

The Laplace transform of some  $f(t)$ , a function of time, is defined as

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t) e^{-st} dt \quad (1.139)$$

where

$$s = \sigma + j\omega \text{ (a complex number)}$$

The strength of the Laplace transform is that it converts linear differential equations with constant coefficients into algebraic equations in the  $s$ -domain. All that remains to do is to take the inverse transform of the explicit solutions to return to the time domain. Although the applications at the school will consider time as the independent variable, a linear differential equation with any independent variable (such as distance) may be solved by Laplace transforms.

There are several important properties of the Laplace transform which should be included in this discussion.

In the general case it can be shown that

$$L \left\{ \frac{d^n f(t)}{dt^n} \right\} = s^n F(s) - \left( s^{n-1} f(0) + s^{n-2} \frac{df(0)}{dt} + \dots + \frac{d^{n-1} f(0)}{dt^{n-1}} \right) \quad (1.140)$$

It is obvious that for quiescent systems (that is, initial conditions zero)'

$$L \left\{ \frac{d^n f(t)}{dt^n} \right\} = s^n F(s) \quad (1.141)$$

This result enables us to write down transfer function by inspection.

Another significant transform is that of an indefinite integral.

In the general case

$$L \left\{ \int \int \int \dots f(t) dt^n \right\} = \frac{F(s)}{s^n} + \frac{f(t) dt}{s^n} \Big|_{t=0+} + \frac{f(t) dt}{s^{n-1}} \Big|_{t=0+} + \dots$$

Equation 1.140 allows us to transform Integro-differential equations such as those arising in electrical engineering.

For the case where all integrals of  $f(t)$  evaluated at  $0+$  are zero, our transform becomes

$$L \left\{ \int \int \int \dots f(t) dt^n \right\} = \frac{F(s)}{s^n} \quad (1.143)$$

A third useful property of the Laplace transform arises if we consider the Laplace transform of the product of some exponential and any other function of time.

$$L \left\{ e^{-at} f(t) \right\} = \int_0^{\infty} e^{-at} f(t) e^{-st} dt = \int_0^{\infty} f(t) e^{-(s+a)t} dt \quad (1.144)$$

It is apparent that this is the same form as the transform of  $f(t)$ , except that the transformed independent variable is  $(s + a)$  rather than  $s$ . We conclude therefore that

$$L \{ e^{-at} f(t) \} = L \{ f(t) \} \Big|_{(s \rightarrow s + a)} = F(s + a) \quad (1.145)$$

It is important to note at this point, that the transform of the product of two functions of time is not equal to the product of the individual transforms. In symbolic form,

$$L \{ f(t) g(t) \} \neq F(s) G(s) \quad (1.146)$$

The  $L \{ f(t) g(t) \}$  must be solved for directly by the definition of the Laplace transform.

The last property we will consider is the Laplace transform of a pure time delay. A pure time delay of the function  $f(t)$  can be represented mathematically as

$$f(t - a) u(t - a) \quad (1.147)$$

where  $a$  is the length of delay and  $u(t - a)$  is the unit step defined as

$$u(t - a) = \begin{cases} 1, & (t - a) > 0 \\ 0, & (t - a) < 0 \end{cases}$$

For such a time delay

$$L \{ f(t - a) u(t - a) \} = e^{-as} L \{ f(t) \} \quad (1.148)$$

We shall now demonstrate the usefulness of the Laplace transform by solving several example problems.

#### EXAMPLE

Solve the given equation for  $x(t)$ ,

$$\dot{x} + 2x = 1 \quad (1.149)$$

when  $x(0) = 1$ .

By Laplace

$$L\{\dot{x}\} = sX(s) - x(0)$$

$$L\{2x\} = 2X(s)$$

$$L\{1\} = \frac{1}{s}$$

Thus

$$(s + 2) X(s) = \frac{1}{s} + 1$$

$$X(s) = \frac{s + 1}{s(s + 2)} = \frac{A}{s} + \frac{B}{s + 2}$$

Solving,

$$A = 1/2$$

and

$$B = 1 - 1/2 = 1/2$$

$$X(s) = \frac{1/2}{s} + \frac{1/2}{s + 2}$$

Inverse transforming gives

$$x(t) = 1/2 - 1/2 e^{-2t} \quad (1.150)$$

EXAMPLE

Given

$$\dot{x} + 2x = \sin t, \quad x(0) = 5 \quad (1.151)$$

solve for  $x(t)$ .

Taking the transform of 1.151

$$sX(s) - x(0) + 2X(s) = \frac{1}{s^2 + 1}$$

and

$$X(s) = \frac{1}{(s^2 + 1)(s + 2)} + \frac{5}{s + 2} \quad (1.152)$$

Expanding the first term on the right side of the equation gives

$$\frac{1}{(s^2 + 1)(s + 2)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 2}$$

Taking the common denominator and equating numerators gives

$$1 = (As + B)(s + 2) + C(s^2 + 1)$$

Substituting values of  $s$  leads to

$$A = -1/5$$

$$B = 2/5$$

$$C = 1/5$$

and substituting back into 1.152 gives

$$X(s) = \frac{-1/5 s}{s^2 + 1} + \frac{2/5}{s^2 + 1} + \frac{1/5}{s + 2} + \frac{5}{s + 2}$$

Inverse transforming gives our solution

$$x(t) = -1/5 \cos t + 2/5 \sin t + 5 i/5 e^{-2t} \quad (1.153)$$

EXAMPLE

Given

$$\ddot{x} + 5\dot{x} + 6x = 3e^{-3t}, \quad x(0) = \dot{x}(0) = 1 \quad (1.154)$$

solve for  $x(t)$ .

Taking the transform of 1.154

$$s^2 X(s) - sx(0) - \dot{x}(0) + 5sX(s) - 5x(0) + 6X(s) = \frac{3}{s + 3}$$

or

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)(s^2 + 5s + 6)}$$

Factoring the denominator,

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)(s + 2)(s + 3)} \quad (1.155)$$

$$= \frac{s^2 + 9s + 21}{(s + 3)^2 (s + 2)} \quad (1.156)$$

$$= \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+2} \quad (1.157)$$

Finding the common denominator of 1.157, and setting the resultant numerator equal to the numerator of 1.156,

$$s^2 + 9s + 21 = A(s+3)(s+2) + B(s+2) + C(s+3)^2$$

which can be solved easily for

$$A = -6$$

$$B = -3$$

$$C = 7$$

Now  $X(s)$  is given by

$$X(s) = \frac{-6}{s+3} - \frac{3}{(s+3)^2} + \frac{7}{s+2}$$

which can be transformed to

$$x(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t} \quad (1.158)$$

#### EXAMPLE

Given

$$\ddot{x} + 2\dot{x} + 10x = 3t + 6/10$$

$$x(0) = 3$$

$$\dot{x}(0) = -27/10$$

solve for  $x(t)$ .

Transforming 1.159 and solving for  $X(s)$  gives

$$X(s) = \frac{3s^3 + 3.3s^2 + 0.6s + 3}{s^2(s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 2s + 10}$$

where

$$A = 0$$

$$B = 0.3$$

$$C = 3$$

$$D = 3$$

Thus,

$$X(s) = \frac{0.3}{s^2} + \frac{3s + 3}{s^2 + 2s + 10} \quad (1.160)$$

To make our inverse transforming a bit easier, let's rewrite 1.160 as

$$X(s) = \frac{0.3}{s^2} + 3 \left[ \frac{(s + 1)}{(s + 1)^2 + 3^2} \right] \quad (1.161)$$

which is readily transformable to

$$x(t) = 0.3t + 3e^{-t} \cos 3t \quad (1.162)$$

PROBLEMS: Set IV, B, page 1.106.

### ● 1.5.5 TRANSFER FUNCTION

Before beginning simultaneous differential equations, we shall define the transfer function of a system. Consider the following equation with initial conditions as shown.

$$a\ddot{x} + b\dot{x} + cx = f(t) \quad (1.163)$$

$$x(0) = \dot{x}(0) = 0$$

If we take the Laplace transform of 1.163, we get

$$as^2X(s) + bsX(s) + cX(s) = F(s) \quad (1.164)$$

or

$$\frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c}$$

Since equation 1.163 represents a system whose input is  $f(t)$  and whose output is  $x(t)$ , we shall define

$$X(s) = \text{output transform}$$

$$F(s) = \text{input transform}$$

We can then define the transfer function of the system, TF, as

$$TF = \frac{X(s)}{F(s)} \quad (1.165)$$

For our example,

$$TF = \frac{1}{as^2 + bs + c} \quad (1.167)$$

Note that the denominator of the transfer function is algebraically the same as the characteristic equation of 1.163. We have already seen, in paragraph 1.6.1 on operator notation, that the characteristic equation completely defines the transient solution, and that the total solution is only altered by the effect of the particular solution due to the input (or forcing function). Thus, from a physical standpoint, the transfer function completely characterizes a linear system.

The transfer function has several properties which we wish to exploit. Suppose that we have two systems characterized by the differential equations

$$a\ddot{x} + b\dot{x} + cx = f(t) \quad (1.168)$$

and

$$d\ddot{y} + e\dot{y} + gy = x(t) \quad (1.169)$$

From the equations it can be seen that the first system has an input  $f(t)$ , and an output  $x(t)$ . The second system has an input  $x(t)$  and an output  $y(t)$ . If we take Laplace transforms at 1.168 and 1.169 we get (assuming all initial conditions are equal to zero)

$$(as^2 + bs + c) X(s) = F(s) \quad (1.170)$$

and

$$(ds^2 + es + g) Y(s) = X(s) \quad (1.171)$$

Finding the transfer functions,

$$TF_1 = \frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c} \quad (1.171)$$

$$TF_2 = \frac{Y(s)}{X(s)} = \frac{1}{ds^2 + es + g} \quad (1.173)$$

Now, both of these systems can be represented schematically as shown in figure 1.16.

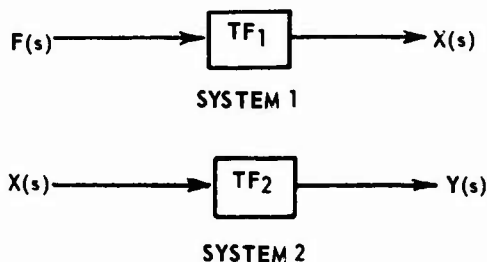


Figure 1.16

Suppose that we now wish to find the output,  $y(t)$ , of system 2 due to the input,  $f(t)$ , of system 1. Our first inspiration might tell us that the logical thing to do is to find  $x(t)$ , but this is not necessary. We can "link" the two systems using the transfer functions, as shown in figure 1.17.

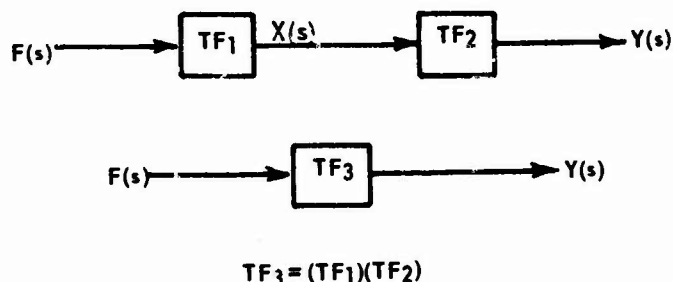


Figure 1.17

The solution we seek,  $y(t)$ , is then given by the inverse transform of  $Y(s)$ , or

$$Y(s) = \begin{bmatrix} TF_3 \end{bmatrix} F(s) \quad (1.174)$$

or

$$Y(s) = \begin{bmatrix} TF_1 \end{bmatrix} \begin{bmatrix} TF_2 \end{bmatrix} F(s) \quad (1.175)$$

This method of solution can be logically extended to include any number of systems we desire.

## ■ 1.6 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATION

In many physical problems, the mathematical description of the system can most conveniently be written as simultaneous differential equations with constant coefficients. The basic procedure for solving a system of  $n$  ordinary differential equations in  $n$  dependent variables consists in obtaining a set of equations from which all but one of the dependent variables, say  $x$ , can be eliminated. The equation resulting from the elimination is then solved for the variable  $x$ . Each of the other dependent variables is then obtained in a similar manner.

We shall consider two procedures for solution of simultaneous linear differential equations, using determinants.

Consider the system

$$2 \frac{dx}{dt} + \frac{dy}{dt} - 4x - y = e^t \quad (1.177)$$

$$\frac{dx}{dt} + 3x + y = 0 \quad (1.178)$$

Using operator notation, they become

$$2(p - 2)x + (p - 1)y = e^t \quad (1.179)$$

$$(p + 3)x + y = 0 \quad (1.180)$$

### ● 1.6.1 SOLUTION BY MEANS OF DETERMINANTS AND OPERATOR NOTATION

Recall that for a determinant of second order the value of the determinant is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \quad (1.176)$$

And then rewrite these equations in the following form

$$p_1 x + p_2 y = f_1(t) \quad (1.181)$$

$$p_3 x + p_4 y = f_2(t) \quad (1.182)$$

where the p's denote the polynomial operators which act on x and y.

Our solution for x can be given by Cramer's rule

$$\begin{vmatrix} p_1 & p_2 \\ p_3 & p_4 \end{vmatrix} x = \begin{vmatrix} f_1(t) & p_2 \\ f_2(t) & p_4 \end{vmatrix} \quad (1.183)$$

and our solution for y can be expressed as

$$\begin{vmatrix} p_1 & p_2 \\ p_3 & p_4 \end{vmatrix} y = \begin{vmatrix} p_1 & f_1(t) \\ p_3 & f_2(t) \end{vmatrix} \quad (1.184)$$

To solve the system given by equations 1.179 and 1.180, we write these equations in determinant form

$$\begin{vmatrix} 2(p - 2) & (p - 1) \\ (p + 3) & 1 \end{vmatrix} x = \begin{vmatrix} e^t & (p - 1) \\ 0 & 1 \end{vmatrix} \quad (1.185)$$

which is expanded to

$$(p^2 + 1)x = -e^t \quad (1.186)$$

giving a solution

$$x(t) = c_1 \cos t + c_2 \sin t - 1/2 e^t \quad (1.187)$$

Solving for y,

$$\begin{vmatrix} 2(p-2) & (p-1) \\ (p+3) & 1 \end{vmatrix} y = \begin{vmatrix} 2(p-2) & e^t \\ (p+3) & 0 \end{vmatrix} \quad (1.188)$$

which can be expanded to

$$(p^2 + 1)y = 4e^t \quad (1.189)$$

giving a solution

$$y(t) = c_3 \cos t + c_4 \sin t + 2e^t \quad (1.190)$$

We know by examining 1.187 and 1.190 that extraneous constants are present, and to eliminate them we substitute back into equation 1.178 and see that

$$(c_2 + 3c_1 + c_3) \cos t + (3c_2 - c_1 + c_4) \sin t = 0 \quad (1.191)$$

Since 1.191 must hold for all values of t, the terms in parenthesis must vanish, giving

$$c_3 = -(3c_1 + c_2)$$

and

$$c_4 = c_1 - 3c_2$$

When these values are substituted in 1.190, we obtain the general solution.

## ● 1.6.2 SOLUTION BY MEANS OF LAPLACE TRANSFORMS

A very effective means of handling simultaneous linear differential equations is to take the Laplace transform of the set of equations and reduce the problem to a set of algebraic equations which can be solved explicitly for the dependent variable in s. This method is demonstrated below.

Given the set of equations

$$3 \frac{d^2 x}{dt^2} + x + \frac{d^2 y}{dt^2} + 3y = f(t) \quad (1.192)$$

$$2 \frac{d^2 x}{dt^2} + x + \frac{d^2 y}{dt^2} + 2y = g(t) \quad (1.193)$$

where  $x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$ , find  $x(t)$  and  $y(t)$ . Taking the Laplace transform of this system yields

$$(3s^2 + 1) X(s) + (s^2 + 3) Y(s) = F(s) \quad (1.194)$$

$$(2s^2 + 1) X(s) + (s^2 + 2) Y(s) = G(s) \quad (1.195)$$

From the previous section, we can solve for  $X(s)$  by rewriting these equations in determinant form, again by Cramer's rule

$$\begin{vmatrix} (3s^2 + 1) & (s^2 + 3) \\ (2s^2 + 1) & (s^2 + 2) \end{vmatrix} X(s) = \begin{vmatrix} F(s) & (s^2 + 3) \\ G(s) & (s^2 + 2) \end{vmatrix} \quad (1.196)$$

Since we are using Laplace transforms instead of operators, however, we can take this equation one step further. We can now solve explicitly for  $X(s)$ , giving us

$$X(s) = \frac{\begin{vmatrix} F(s) & (s^2 + 3) \\ G(s) & (s^2 + 2) \end{vmatrix}}{\begin{vmatrix} (3s^2 + 1) & (s^2 + 3) \\ (2s^2 + 1) & (s^2 + 2) \end{vmatrix}} \quad (1.197)$$

In a similar manner,

$$Y(s) = \frac{\begin{vmatrix} (3s^2 + 1) & F(s) \\ (2s^2 + 1) & G(s) \end{vmatrix}}{\begin{vmatrix} (3s^2 + 1) & (s^2 + 3) \\ (2s^2 + 1) & (s^2 + 2) \end{vmatrix}} \quad (1.198)$$

For the particular inputs  $f(t) = t$  and  $g(t) = 1$ ,

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s^2} & (s^2 + 3) \\ \frac{1}{s} & (s^2 + 2) \end{vmatrix}}{(s^4 - 1)} = \frac{-s^3 + s^2 - 3s + 2}{s^2(s^4 - 1)} \quad (1.199)$$

Expanded as a partial fraction

$$X(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{(s^2 + 1)} + \frac{E}{s - 1} + \frac{F}{(s + 1)} = \frac{-s^3 + s^2 - 3s + 2}{s^2(s^4 - 1)} \quad (1.200)$$

Solving for A, B, etc., we have

$$X(s) = \frac{-2}{s^2} + \frac{3}{s} + \frac{1/2 - s}{s^2 + 1} - \frac{7/4}{s + 1} - \frac{1/4}{s - 1} \quad (1.201)$$

which yields a solution

$$x(t) = -2t + 3 - 7/4e^{-t} - 1/4e^t + 1/2 \sin t - \cos t \quad (1.202)$$

A similar approach will obtain the solution for  $y(t)$ .

In the case of three simultaneous differential equations, the application of Laplace will yield the proper solutions.

$$P_1(s) X(s) + P_2(s) Y(s) + P_3(s) Z(s) = F_1(s) \quad (1.203)$$

$$Q_1(s) X(s) + Q_2(s) Y(s) + Q_3(s) Z(s) = F_2(s) \quad (1.204)$$

$$R_1(s) X(s) + R_2(s) Y(s) + R_3(s) Z(s) = F_3(s) \quad (1.205)$$

where

$$X(s) = \frac{\begin{vmatrix} F_1 & P_2 & P_3 \\ F_2 & Q_2 & Q_3 \\ F_3 & R_2 & R_3 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix}} \quad (1.206)$$

$Y(s)$  and  $Z(s)$  will have similar forms.

PROBLEMS: Set V, page 1.119.

Table II  
LAPLACE TRANSFORMS

	$F(s)$	$f(t)$
1.	$\int_0^{\infty} e^{-st} f(t) dt$	$f(t)$
2.	$s^n F(s) - s^{n-1} f(0+) - s^{n-2} f'(0+) - \dots - f^{(n-1)}(0+)$	$f^{(n)}(t)$
3.	$\frac{1}{s}$	1
4.	$\frac{1}{s^2}$	$t$
5.	$\frac{n!}{s^{n+1}} \quad (n = 1, 2, \dots)$	$t^n$
6.	$\frac{1}{s + a}$	$e^{-at}$
7.	$\frac{1}{(s + a)^2}$	$te^{-at}$
8.	$\frac{n!}{(s + a)^{n+1}} \quad (n = 1, 2, \dots)$	$t^n e^{-at}$
9.	$\frac{1}{(s + a)(s + b)} \quad a \neq b$	$\frac{1}{b - a} (e^{-at} - e^{-bt})$
10.	$\frac{s}{(s + a)(s + b)} \quad a \neq b$	$\frac{1}{a - b} (ae^{-at} - be^{-bt})$
11.	$\frac{1}{(s + a)(s + b)(s + c)}$	$\frac{(b - c)e^{-at} - (a - c)e^{-bt} + (a - b)e^{-ct}}{(a - b)(b - c)(a - c)}$
12.	$\frac{a}{s^2 + a^2}$	$\sin at$

Table II (Concluded)

	$F(s)$	$f(t)$
13.	$\frac{s}{s^2 + a^2}$	$\cos at$
14.	$\frac{a^2}{s(s^2 + a^2)}$	$1 - \cos at$
15.	$\frac{a^3}{s^2(s^2 + a^2)}$	$at - \sin at$
16.	$\frac{2a^3}{(s^2 + a^2)^2}$	$\sin at - at \cos at$
17.	$\frac{2as}{(s^2 + a^2)^2}$	$t \sin at$
18.	$\frac{2as^2}{(s^2 + a^2)^2}$	$\sin at + at \cos at$
19.	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	$t \cos at$
20.	$\frac{(b^2 - a^2)s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\cos at - \cos bt$
21.	$\frac{b}{(s + a)^2 + b^2}$	$e^{-at} \sin bt$
22.	$\frac{s + a}{(s + a)^2 + b^2}$	$e^{-at} \cos bt$

## 1.7 PROBLEM SET I

1. Solve for  $y$ .

a.  $\frac{dv}{dx} = x^4 + 4x + \sin 6x$

b.  $\frac{d^2y}{dx^2} = e^{-x} + \sin \omega x$

c.  $\frac{d^3y}{dx^3} = x^5$

d.  $y \frac{dy}{dx} + 3x^2 = 0$

e.  $(x-1)^2 y dx + x^2 (y+1) dy = 0$

2. Test for exactness and solve if exact.

a.  $(y^2 - x)dx + (x^2 - y)dy = 0$

b.  $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$

c.  $(2xy^4 e^y + 2xy^3 + y)dx + (x^2 y^4 e^y - y^2 y^2 - 3x)dy = 0$

d. Multiply c. by  $1/y^4$ .

3. Solve for  $y_t$  using operator notation.

a.  $5y' + 6y = 0$

b.  $y''' - 5y'' - 24y' = 0$

c.  $y'' + 12y' + 36y = 0$

d.  $y''' + 25y'' = 0$

e.  $y'' + 4y' + 13y = 0$

# ■ 1.8 SOLUTION TO PROBLEM SET I

$$\underline{1a.} \quad \frac{dy}{dx} = x^4 + 4x + \sin 6x$$

By direct integration

$$\int dy = \int (x^4 + 4x + \sin 6x) dx + C$$

$$y = \frac{x^5}{5} + 2x^2 - \frac{\cos 6x}{6} + C$$

$$\underline{1b.} \quad \frac{d^2y}{dx^2} = e^{-x} + \sin \omega x$$

By direct integration

$$\int dy' = \int (e^{-x} + \sin \omega x) dx + C_1$$

$$y' = -e^{-x} - \frac{\cos \omega x}{\omega} + C_1$$

$$\int dy = \int (-e^{-x} - \frac{\cos \omega x}{\omega} + C_1) dx + C_2$$

$$y = e^{-x} - \frac{\sin \omega x}{\omega^2} + C_1 x + C_2$$

1c.  $y''' = x^5$

By direct integration

$$\int dy'' = \int x^5 dx + C_1$$

$$y'' = \frac{x^6}{6} + C_1$$

$$\int dy' = \int \left( \frac{x^6}{6} + C_1 \right) dx + C_2$$

$$y' = \frac{x^7}{42} + C_1 x + C_2$$

$$\int dy = \int \left( \frac{x^7}{42} + C_1 x + C_2 \right) dx + C_3$$

$$y = \frac{x^8}{336} + \frac{C_1 x^2}{2} + C_2 x + C_3$$

1d.  $y \frac{dy}{dx} + 3x^2 = 0$

Separate Variables

$$\int y dy = \int -3x^2 dx + C$$

$$\frac{y^2}{2} = -x^3 + C$$

1e.  $(x - 1)^2 y dx + x^2 (y + 1) dy = 0$

Separate Variables

$$\frac{y + 1}{y} dy = - \frac{(x - 1)^2}{x^2} dx$$

$$\int (1 + \frac{1}{y}) dy = - \int (\frac{x^2 - 2x + 1}{x^2}) dx + C$$

$$\int (1 + \frac{1}{y}) dy = - \int (1 - \frac{2}{x} + \frac{1}{x^2}) dx + C$$

$$y + \ln y = -x + 2 \ln x + \frac{1}{x} + C$$


---

2a.  $(y^2 - x) dx + (x^2 - y) dy = 0$

$$M = (y^2 - x) \qquad N = x^2 - y$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \implies \text{Not Exact}$$


---

2b.  $(2x^3 + 3y) dx + (3x + y - 1) dy = 0$

$$M = 2x^3 + 3y \qquad N = 3x + y - 1$$

$$\frac{\partial M}{\partial y} = 3 \quad \frac{\partial N}{\partial x} = 3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{Exact}$$

$$\int_a^x (2x^3 + 3y) dx + \int_b^y (3a + y - 1) dy = k$$

$$\left( \frac{1}{2}x^4 + 3xy \right) \Big|_a^x + \left( 3ay + \frac{1}{2}y^2 - y \right) \Big|_b^y = k$$

$$\left( \frac{1}{2}x^4 + 3xy \right) - \left( \frac{1}{2}a^4 + 3ay \right) + \left( 3ay + \frac{1}{2}y^2 - y \right) - \left( 3ab + \frac{1}{2}b^2 - b \right) = k$$

$$\frac{1}{2}x^4 + 3xy + \frac{1}{2}y^2 - y = k + \frac{1}{2}a^4 + 3ab + \frac{1}{2}b^2 - b = k$$

$$\frac{1}{2}x^4 + 3xy + \frac{1}{2}y^2 - y = k$$

2c.  $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$$M = 2xy^4e^y + 2xy^3 + y \quad N = x^2y^4e^y - x^2y^2 - 3x$$

$$\frac{\partial M}{\partial y} = 2x(4y^3e^y + y^4e^y) + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{Not Exact}$$

2d.  $\left( 2xe^y + \frac{2x}{y} + \frac{1}{y^3} \right) dx + \left( x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} \right) dy = 0$

$$M = \left( 2xe^y + \frac{2x}{y} + \frac{1}{y^3} \right) \quad N = \left( x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} \right)$$

$$\frac{\partial M}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

$$\frac{\partial N}{\partial x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

$$\frac{\partial N}{\partial y} = \frac{\partial M}{\partial x} \Rightarrow \text{Exact}$$

$$\int_a^x (2xe^y + \frac{2x}{y} + \frac{1}{y^3}) dx + \int_b^y (a^2e^y - \frac{a^2}{y^2} - \frac{3a}{y^4}) dy = k$$

$$(x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}) \Big|_a^x + (a^2e^y + \frac{a^2}{y} + \frac{a}{y^3}) \Big|_b^y = k$$

$$(x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}) - (a^2e^y + \frac{a^2}{y} + \frac{a}{y^3}) + (a^2e^y + \frac{a^2}{y} + \frac{a}{y^3}) - (a^2e^b + \frac{a^2}{b} + \frac{a}{b^3}) = k$$

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = k + a^2e^b + \frac{a^2}{b} + \frac{a}{b^3} = k$$

3a.  $5y' + 6y = 0$

$$5p + 6 = 0$$

$$p = -\frac{6}{5}$$

$$y_t = Ce^{-6/5x}$$

$$\underline{3b.} \quad y''' - 5y'' - 24y' = 0$$

$$p^3 - 5p^2 - 24p = 0$$

$$p(p^2 - 5p - 24) = 0$$

$$p(p - 8)(p + 3) = 0$$

$$p = 0, 8, -3$$

$$y_t = C_1 + C_2 e^{8x} + C_3 e^{-3x}$$


---

$$\underline{3c.} \quad y'' + 12y' + 36y = 0$$

$$p^2 + 12p + 36 = 0$$

$$(p + 6)(p + 6) = 0$$

$$p = -6, -6$$

$$y_t = C_1 e^{-6x} + C_2 x e^{-6x}$$


---

3d.  $y^{IV} + 25y'' = 0$

$$p^4 + 25p^2 = 0$$

$$p^2 (p^2 + 25) = 0$$

$$p = 0, 0, \pm 5j$$

$$y_t = C_1 + C_2 x + C_3 \sin (5x + \emptyset)$$

or

$$y_t = C_1 + C_2 x + C_4 \sin 5x + C_5 \cos 5x$$


---

3e.  $y'' + 4y' + 13y = 0$

$$p^2 + 4p + 13 = 0$$

$$(p + 2)^2 + 9 = 0$$

$$p + 2 = \pm 3j$$

$$p = -2 \pm 3j$$

$$y_t = C_1 e^{-2x} \sin (3x + \emptyset)$$

or

$$y_t = e^{-2x} (C_2 \sin 3x + C_3 \cos 3x)$$


---

## ■ 1.9 PROBLEM SET II

Given:

$$1. \ddot{y} + 36\ddot{y} = 6 + t$$

$$2. \ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$$

$$3. \ddot{y} + 4\dot{y} + 4y = \cos t$$

$$4. 2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$$

$$5. 3\dot{x} + 2x = -4e^{-2t}$$

Find:

- The transient solution.
- The particular solution.
- Substitute the following boundary conditions to eliminate arbitrary constants.

$$1) \text{ For problem 1 above } \ddot{y}(0) = \frac{1}{36}, \ddot{y}(0) = \frac{1}{6}, \dot{y}(0) = y(0) = 0$$

$$2) \text{ For problem 2 } \dot{y}(0) = -6, y(0) = 1$$

$$3) \text{ For problem 3 } y(0) = \frac{28}{25}, \dot{y}(0) = -\frac{104}{25}$$

$$4) \text{ For problem 4 } \dot{x}(0) = -\frac{27}{10}, x(0) = 3$$

$$5) \text{ For problem 5 } x(3) = -0.14$$

- d. Find  $\omega_n$  ,  $\omega_d$  ,  $\zeta$  ,  $\tau$  where applicable.
- e. Describe the system damping where applicable (i.e. underdamped, overdamped, etc.).
- f. SKETCH the total response (i.e. total solution).

# ■ 1.10 SOLUTION TO PROBLEM SET II

$$1. \quad \ddot{y} + 36\dot{y} = 6 + t$$

$$a. \quad p^4 + 36p^2 = 0$$

$$p^2 (p^2 + 36) = 0$$

$$p = 0, 0, \pm 6j$$

$$y_t = C_1 + C_2 t + C_3 \sin(6t + \phi)$$

or

$$y_t = C_1 + C_2 t + C_4 \sin 6t + C_5 \cos 6t$$

b. Assume

$$y_p = At + B$$

Checking the transient solution we see both of these same terms. We must multiply by the independent variable until we do not duplicate terms in the transient solution.

$$y_p = At^3 + Bt^2$$

$$\dot{y}_p = 3At^2 + 2Bt$$

$$\ddot{y}_p = 6At + 2B$$

$$\dots y_p = 6A$$

$$\dots y_p = 0$$

Substituting into the original differential equation,

$$0 + 36(6At + 2B) = 6 + t$$

$$216At + 72B = 6 + t$$

Equating like coefficients,

$$216A = 1$$

$$A = \frac{1}{216}$$

$$72B = 6$$

$$B = \frac{1}{12}$$

$$y_p = \frac{t^3}{216} + \frac{t^2}{12}$$

$$c) \quad y = y_t + y_p$$

$$(1) \quad y = C_1 + C_2 t + C_3 \sin(6t + \emptyset) + \frac{t^3}{216} + \frac{t^2}{12}$$

$$(2) \quad y = C_1 + C_2 t + C_4 \sin 6t + C_5 \cos 6t + \frac{t^3}{216} + \frac{t^2}{12}$$

Using (1)

$$\dot{y} = C_2 + 6C_3 \cos(6t + \emptyset) + \frac{t^2}{72} + \frac{t}{6}$$

$$\ddot{y} = -36C_3 \sin(6t + \emptyset) + \frac{t}{36} + \frac{1}{6}$$

$$\ddot{\ddot{y}} = -216C_3 \cos(6t + \emptyset) + \frac{1}{36}$$

Substituting boundary conditions.

$$\ddot{y}(0) = -216C_3 \cos \theta + \frac{1}{36} = \frac{1}{36}$$

$$C_3 \cos \theta = 0$$

$$\ddot{y}(0) = -36C_3 \sin \theta + \frac{1}{6} = \frac{1}{6}$$

$$C_3 \sin \theta = 0$$

$$C_3 \sin \theta = C_3 \cos \theta$$

$$\sin \theta = \cos \theta$$

$$\theta = 45^\circ$$

$$C_3 = 0$$

$$C_2 = 0$$

$$C_1 = 0$$

$$y = \frac{t^3}{216} + \frac{t^2}{12}$$

Using (2)

$$y = C_1 + C_2 t + C_4 \sin 6t + C_5 \cos 6t + \frac{t^3}{216} + \frac{t^2}{12}$$

$$\dot{y} = C_2 + 6C_4 \cos 6t - 6C_5 \sin 6t + \frac{t^2}{12} + \frac{t}{6}$$

$$\ddot{y} = -36C_4 \sin 6t - 36C_5 \cos 6t + \frac{t}{3} + \frac{1}{6}$$

$$\ddot{y} = -216C_4 \cos 6t + 216C_5 \sin 6t + \frac{1}{36}$$

$$\ddot{y}(0) = -216c_4 + \frac{1}{36} = \frac{1}{36}$$

$$c_4 = 0$$

$$\ddot{y}(0) = -36c_5 + \frac{1}{6} = \frac{1}{6}$$

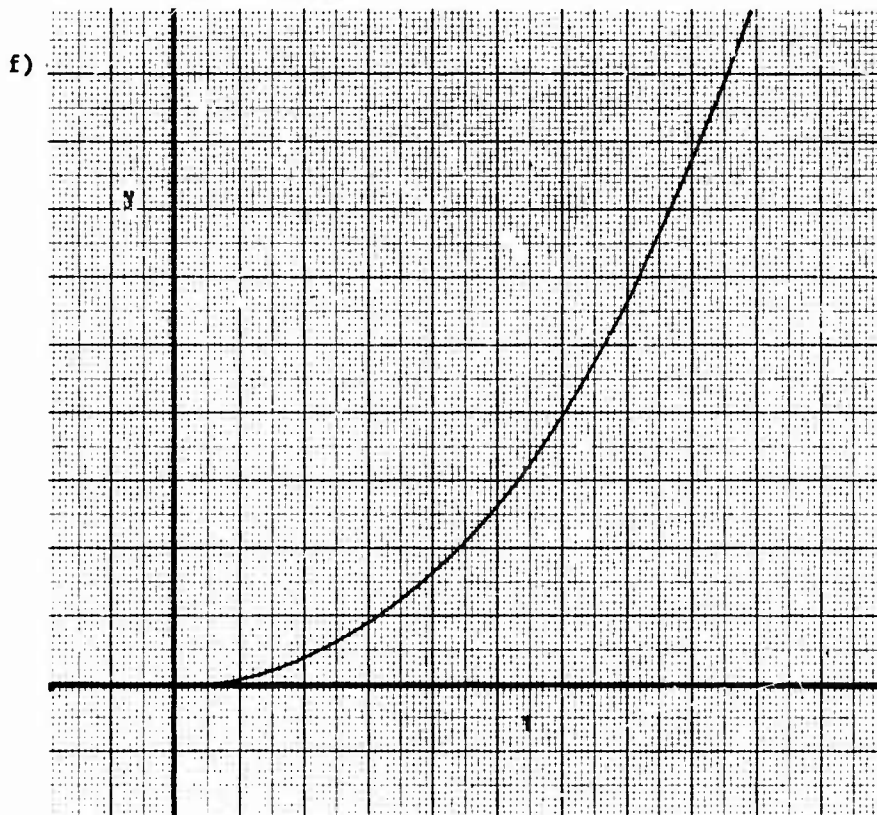
$$c_5 = 0$$

$$\dot{y}(0) = c_2 = 0$$

$$y(0) = c_1 = 0$$

$$\therefore y = \frac{t^3}{216} + \frac{t^2}{12}$$

Numbers d) and e) do not apply.



$$2. \ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$$

$$a) p^2 + 5p + 6 = 0$$

$$(p + 3)(p + 2) = 0$$

$$p = -3, -2$$

$$y_t = C_1 e^{-3t} + C_2 e^{-2t}$$

b) Assume  $y_p = Ae^{-3t}$  (forcing function and all its derivatives)

Cross check with  $y_t$ . Since  $y_p$  appears in  $y_t$  multiply by  $t$  in order to eliminate the duplication.

$$y_p = Ate^{-3t}$$

$$\dot{y}_p = -3Ate^{-3t} + Ae^{-3t}$$

$$\ddot{y}_p = 9Ate^{-3t} - 3Ae^{-3t} - 3Ae^{-3t}$$

$$= 9Ate^{-3t} - 6Ae^{-3t}$$

Substituting into the original D.E.

$$e^{-3t} [(9At - 6A) + 5(A - 3At) + 6At] = 3e^{-3t}$$

$$e^{-3t} (-A) = 3e^{-3t}$$

$$A = -3$$

$$y_p = -3te^{-3t}$$

$$c) \quad y = C_1 e^{-3t} + C_2 e^{-2t} - 3te^{-3t}$$

$$\dot{y} = -3C_1 e^{-3t} - 2C_2 e^{-2t} + 9te^{-3t} - 3e^{-3t}$$

Substituting the boundary conditions

$$y(0) = 1 = C_1 + C_2 \Rightarrow C_2 = 1 - C_1$$

$$\dot{y}(0) = -6 = -3C_1 - 2C_2 - 3$$

$$-3 = -3C_1 - 2 + 2C_1$$

$$C_1 = 1$$

$$C_2 = 0$$

$$y = (1 - 3t)e^{-3t}$$

From the homogeneous equation

$$y + 5\dot{y} + 6y = 0$$

$$\omega_n^2 = 6$$

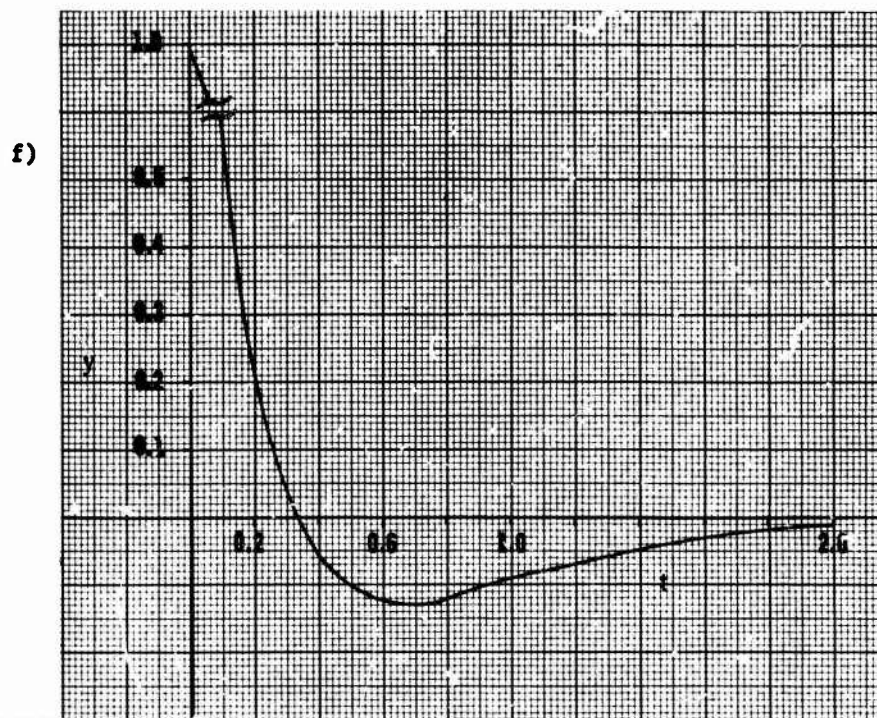
$$\omega_n = \sqrt{6}$$

$$2\zeta\omega_n = 5$$

$$\zeta = \frac{5}{2\sqrt{6}} = 1.02$$

$$\omega_d = \text{is undefined}$$

The system is overdamped and non-oscillatory.



3.  $\ddot{y} + 4\dot{y} + 4y = \cos t$

a)  $p^2 + 4p + 4 = 0$

$$(p + 2)^2 = 0$$

$$p = -2, -2$$

$$y_t = C_1 e^{-2t} + C_2 t e^{-2t}$$

b)  $y_p = A_1 \sin t + A_2 \cos t$

$$\dot{y}_p = A_1 \cos t - A_2 \sin t$$

$$\ddot{y}_p = -A_1 \sin t - A_2 \cos t$$

Substituting into our original D.E.

$$(-A_1 - 4A_2 + 4A_1) \sin t + (-A_2 + 4A_1 + 4A_2) \cos t = \cos t$$

$$(3A_1 - 4A_2) \sin t + (4A_1 + 5A_2) \cos t = \cos t$$

Equating like coefficients

$$\begin{array}{rcl} 3A_1 - 4A_2 & = & 0 \\ A_1 & = & \frac{4}{3} A_2 \\ A_1 & = & \frac{4}{25} \end{array} \quad \begin{array}{l} \text{Sub.} \nearrow \\ \text{Sub.} \searrow \end{array} \quad \begin{array}{rcl} 4A_1 + 3A_2 & = & 1 \\ \frac{16}{3} A_2 + 3A_2 & = & 1 \\ A_2 & = & \frac{3}{25} \end{array}$$

$$\therefore y_p = \frac{4}{25} \sin t + \frac{3}{25} \cos t$$

$$y = y_t + y_p$$

$$c) \quad y = C_1 e^{-2t} + C_2 t e^{-2t} + \frac{4}{25} \sin t + \frac{3}{25} \cos t$$

$$y(0) = C_1 + \frac{3}{25} = \frac{28}{25}$$

$$C_1 = 1$$

$$\dot{y}(t) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t} + \frac{4}{25} \cos t - \frac{3}{25} \sin t$$

$$\dot{y}(0) = -2C_1 + C_2 + \frac{4}{25}$$

$$= -\frac{46}{25} + C_2 = -\frac{104}{25}$$

$$C_2 = -\frac{58}{25}$$

$$\therefore y = e^{-2t} - \frac{58}{25} t e^{-2t} + \frac{4}{25} \sin t + \frac{3}{25} \cos t$$

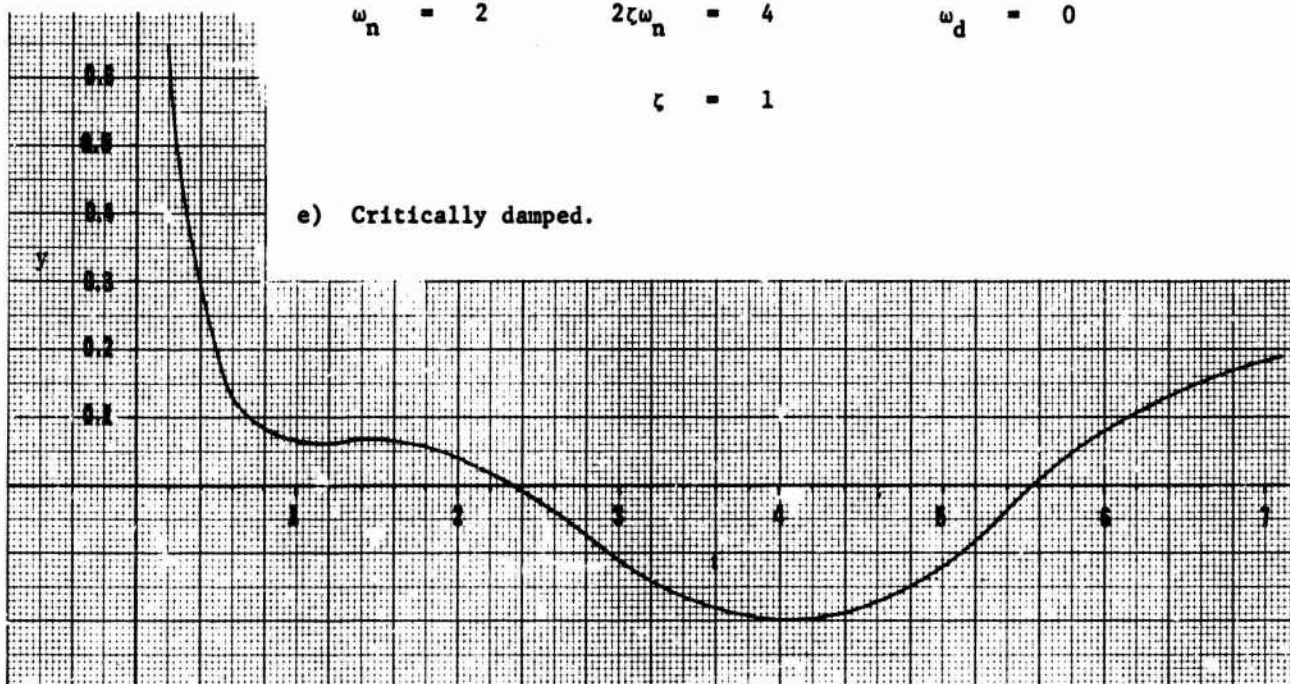
$$d) \ddot{y} + 4\dot{y} + 4 = 0$$

$$\omega_n = 2$$

$$2\zeta\omega_n = 4$$

$$\omega_d = 0$$

$$\zeta = 1$$



e) Critically damped.

$$4. \quad 2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$$

Place in "Standard Form"  $\ddot{x} + 2\dot{x} + 10x = 0$

$$a) \quad \ddot{x} + 2\dot{x} + 10x = 0$$

$$p^2 + 2p + 10 = 0$$

$$(p + 1)^2 + 9 = 0$$

$$p = -1 \pm 3j$$

$$\therefore x_t = e^{-t} (C_1 \sin 3t + C_2 \cos 3t)$$

b) Assume  $x_p = At + B$

$$\dot{x}_p = A$$

$$\ddot{x}_p = 0$$

Substituting

$$0 + 4A + 20At + 20B = 6t + \frac{6}{5}$$

$$20A = 6$$

$$A = \frac{3}{10}$$

$$4A + 20B = \frac{6}{5}$$

$$B = 0$$

$$\therefore x_p = \frac{3}{10} t$$

$$x = x_t + x_p$$

c)  $x(t) = e^{-t} (C_1 \sin 3t + C_2 \cos 3t) + \frac{3}{10} t$

$$\begin{aligned} \dot{x}(t) &= -e^{-t} (C_1 \sin 3t + C_2 \cos 3t) + e^{-t} (3C_1 \cos 3t \\ &\quad - 3C_2 \sin 3t) + \frac{3}{10} \end{aligned}$$

$$x(0) = 3 = C_2$$

$$\begin{aligned} \dot{x}(0) &= -\frac{27}{10} = -C_2 + 3C_1 + \frac{3}{10} \\ -\frac{27}{10} &= -3 + 3C_1 + \frac{3}{10} \end{aligned}$$

$$c_1 = 0$$

$$\therefore x(t) = 3e^{-t} \cos 3t + \frac{3}{10} t$$

$$d) \ddot{x} + 2\dot{x} + 10x = 0$$

$$\omega_n = \sqrt{10}$$

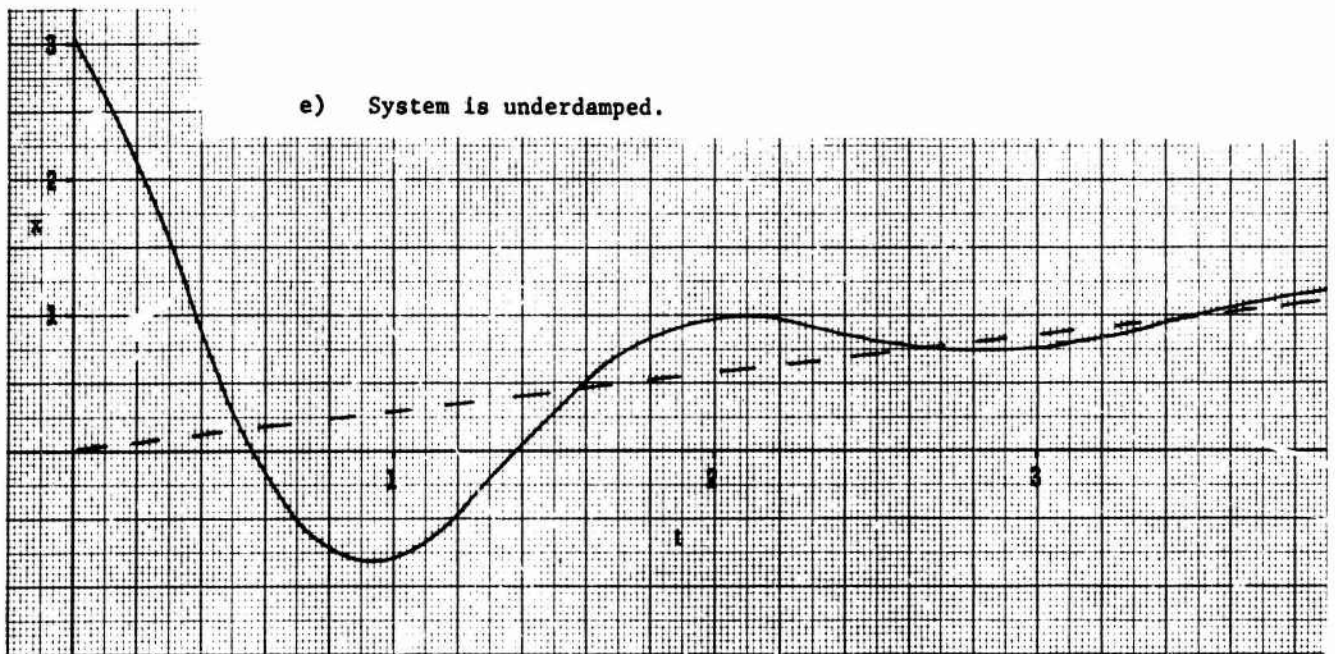
$$2\zeta\omega_n = 2$$

$$\zeta = 1/\sqrt{10}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$\omega_d = 3$$

e) System is underdamped.



5. Given

$$3\dot{x} + 2x = -4e^{-2t}$$

a)  $3p + 2 = 0$

$$p = -2/3$$

$$x_t = C_1 e^{-2t/3}$$

b) Assume

$$x_p = Ae^{-2t}$$

$$\dot{x}_p = -2Ae^{-2t}$$

Substituting

$$-6Ae^{-2t} + 2Ae^{-2t} = -4e^{-2t}$$

$$A = 1$$

$$x_p = e^{-2t}$$

c)  $x = C_1 e^{-2t/3} + e^{-2t}$

Substituting the boundary condition

$$x(3) = -0.14 = C_1(0.14) + .002$$

$$C_1 = \frac{-0.142}{.14} = -1.01 \approx -1$$

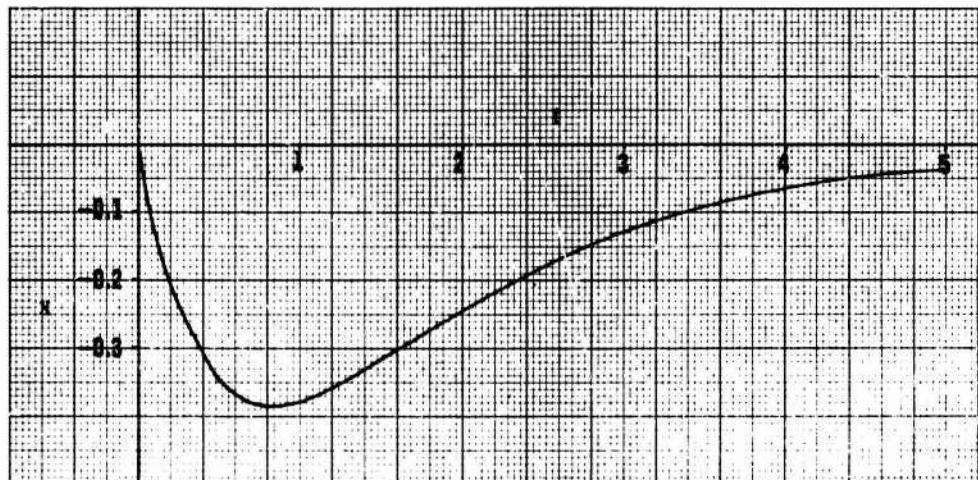
$$x(t) = -e^{-2t/3} + e^{-2t}$$

d) From the transient solution

$$x_t = C_1 e^{-2t/3}$$

$$\tau = 1.5$$

e) N/A



■ 1.11 PROBLEM SET III

Given:  $C_1 \ddot{x} + C_2 \dot{x} + C_3 x = f(t)$

Find:  $X(s)$

When:

	#1	#2	#3	#4	#5
$C_1 =$	3	1	4	2	1
$C_2 =$	1	-2	3	4	6
$C_3 =$	6	5	-1	7	9
$x(0) =$	0	-1	3	-1	2
$\dot{x}(0) =$	0	9	-2	0	-1
$f(t) =$	$\sin 6t$	$e^{-t} \sin 3t$	$t^3 - t \sin 2t$	$6e^{-4t} + 6t \cos 4t$	$\frac{2 - \cos 2t}{4}$

### ■ 1.12 SOLUTION TO PROBLEM SET III

In general for a second order D.E.

$$C_1 \ddot{x} + C_2 \dot{x} + C_3 x = f(t)$$

Take the Laplace Transform

$$C_1 [s^2 X(s) - sx(0) - \dot{x}(0)] + C_2 [sX(s) - x(0)] + C_3 X(s) = F(s)$$

$$(C_1 s^2 + C_2 s + C_3)X(s) = F(s) + C_1 sx(0) + C_1 \dot{x}(0) + C_2 x(0)$$

$$X(s) = \frac{F(s) + C_1 sx(0) + C_1 \dot{x}(0) + C_2 x(0)}{(C_1 s^2 + C_2 s + C_3)}$$

This equation can be used for any second order D.E. with constant coefficients.

1. Given

$$C_1 = 3$$

$$C_2 = 1$$

$$C_3 = 6$$

$$x(0) = \dot{x}(0) = 0$$

$$f(t) = \sin 6t$$

From transform pair 12,  $F(s) = \frac{6}{s^2 + 36}$

$$X(s) = \frac{6}{(s^2 + 36)(3s^2 + s + 6)}$$


---

2. Given:

$$C_1 = 1$$

$$C_2 = -2$$

$$C_3 = 5$$

$$x(0) = -1$$

$$\dot{x}(0) = 9$$

$$f(t) = e^{-t} \sin 3t$$

From transform pair 21,

$$F(s) = \frac{3}{(s+1)^2 + 9} = \frac{3}{s^2 + 2s + 10}$$

$$X(s) = \left[ \frac{\left( \frac{3}{s^2 + 2s + 10} \right) - s + 11}{s^2 - 2s + 5} \right]$$

Obviously, this can be reduced to a simpler form, but that's not necessary for this exercise.

---

3. Given:

$$c_1 = 4$$

$$c_2 = 3$$

$$c_3 = -1$$

$$x(0) = 3$$

$$\dot{x}(0) = -2$$

$$f(t) = t^3 - t \sin 2t$$

From transform pairs 5 and 17,

$$F(s) = \frac{6}{s^4} - \frac{4s}{(s^2 + 4)^2}$$

$$= \frac{6s^4 - 4s^5 + 48s^2 + 96}{s^4(s^2 + 4)^2}$$

$$X(s) = \left[ \frac{\left( \frac{6s^4 - 4s^5 + 48s^2 + 96}{s^4(s^2 + 4)^2} \right) + 12s + 1}{(4s^2 + 3s - 1)} \right]$$

---

4. Given:

$$c_1 = 2$$

$$c_2 = 4$$

$$c_3 = 7$$

$$x(0) = -1$$

$$\dot{x}(0) = 0$$

$$f(t) = 6e^{-4t} + 6t \cos 4t$$

From transform pairs 6 and 19,

$$F(s) = \frac{6}{s+4} + \frac{6(s^2 - 16)}{(s^2 + 16)^2}$$

$$F(s) = \frac{6s^4 + 192s^2 + 1536 + 6s^3 + 24s^2 - 96s - 384}{(s+4)(s^2+16)^2}$$

$$F(s) = \frac{6s^4 + 6s^3 + 216s^2 - 96s + 1152}{(s+4)(s^2+16)^2}$$

$$X(s) = \left[ \frac{F(s) - (2s+4)}{2s^2 + 4s + 7} \right]$$

---

5. Given:

$$C_1 = 1$$

$$C_2 = 6$$

$$C_3 = 9$$

$$x(0) = 2$$

$$\dot{x}(0) = -1$$

$$f(t) = \frac{2 - \cos 2t}{4} = \frac{1}{2} - \frac{1}{4} \cos 2t$$

From transform pairs 3 and 13,

$$F(s) = \frac{1}{2} \left( \frac{1}{s} \right) - \frac{1}{4} \left( \frac{s}{s^2 + 4} \right)$$

$$F(s) = \frac{1}{4} \left[ \frac{2s^2 + 8 - s^2}{s(s^2 + 4)} \right] = \frac{1}{4} \left[ \frac{s^2 + 8}{s(s^2 + 4)} \right]$$

$$X(s) = \left[ \frac{F(s) + (2s + 11)}{s^2 + 6s + 9} \right]$$

### 1.13 PROBLEM SET IV

A. Expand by partial fractions:

$$1) \frac{5s^2 + 29s + 36}{(s+2)(s^2 + 4s + 3)}$$

$$2) \frac{2s^2 + 6s + 5}{(s^2 + 3s + 2)(s+1)}$$

$$3) \frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{(s^3 + 9s)(s^2 + 3s + 3)}$$

$$4) \frac{7s^4 - 3s^3 + 56s^2 - 17s + 107}{(s-1)(s^2 + 4)^2}$$

B. Solve by Laplace:

$$1) \dot{y} + y = 5, \quad y(0) = 0$$

$$2) \dot{y} + y = e^{-3t}, \quad y(0) = 1$$

$$3) \dot{x} + 2x = \sin t, \quad x(0) = 5$$

$$4) \ddot{x} + 2\dot{x} + 9x = 2 \sin 3t, \quad x(0) = \dot{x}(0) = 0$$

$$5) \ddot{x} + 5\dot{x} + 6x = 3e^{-3t}, \quad x(0) = \dot{x}(0) = 1$$

$$6) \ddot{x} + 2\dot{x} + 10x = 3t + \frac{3}{5}, \quad x(0) = 3, \quad \dot{x}(0) = -\frac{27}{10}$$

# ■ 1.14 SOLUTION TO PROBLEM SET IV

A.

$$1) \frac{5s^2 + 29s + 36}{(s+2)(s^2 + 4s + 3)} = \frac{5s^2 + 29s + 36}{(s+2)(s+3)(s+1)}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+1}$$

$$= \frac{A(s+3)(s+1) + B(s+2)(s+1) + C(s+2)(s+3)}{(s+1)(s+2)(s+3)}$$

Setting numerators equal:

$$5s^2 + 29s + 36 = A(s+3)(s+1) + B(s+2)(s+1) + C(s+2)(s+3)$$

$$\text{let } s = -3 : 45 - 87 + 36 = B(-1)(-2)$$

$$2B = -6$$

$$\underline{B = -3}$$

$$\text{let } s = -1 : 5 - 29 + 36 = C(1)(2)$$

$$2C = 12$$

$$\underline{C = 6}$$

$$\text{let } s = -2 : 20 - 58 + 36 = A(1)(-1)$$

$$\underline{A = 2}$$

$$\frac{5s^2 + 29s + 36}{(s+2)(s^2 + 4s + 3)} = \frac{2}{s+2} + \frac{(-3)}{s+3} + \frac{6}{s+1}$$


---

$$2) \frac{2s^2 + 6s + 5}{(s^2 + 3s + 2)(s+1)} = \frac{2s^2 + 6s + 5}{(s+2)(s+1)(s+1)} = \frac{2s^2 + 6s + 5}{(s+2)(s+1)^2}$$

$$= \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$= \frac{A(s+1)^2 + B(s+2)(s+1) + C(s+2)}{(s+2)(s+1)^2}$$

Setting numerators equal:

$$2s^2 + 6s + 5 = A(s+1)^2 + B(s+2)(s+1) + C(s+2)$$

$$\text{let } s = -1 : 2 - 6 + 5 = C$$

$$\underline{C = 1}$$

$$\text{let } s = -2 : 8 - 12 + 5 = A$$

$$\underline{A = 1}$$

$$\text{let } s = 0 : 5 = A + B(2) + C(2)$$

$$5 = 1 + 2B + 2$$

$$\underline{B = 1}$$

$$\frac{2s^2 + 6s + 5}{(s+2)(s+1)^2} = \frac{1}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$3) \quad \frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{(s^3 + 9s)(s^2 + 3s + 3)} = \frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{s(s^2 + 9)(s^2 + 3s + 3)}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + 9} + \frac{Ds + E}{s^2 + 3s + 3}$$

Find the common denominator and set numerators equal:

$$2s^4 + 7s^3 + 27s^2 + 51s + 27 = A(s^2 + 9)(s^2 + 3s + 3) + (Bs + C)(s)(s^2 + 3s + 3) + (Ds + E)(s)(s^2 + 9)$$

$$\text{let } s = 0 : 27 = 27A$$

$$\underline{A = 1}$$

$$\text{let } s = 3j : 162 + (-189j) - 243 + 153j + 27 = (3Bj + C)$$

$$= (3j)(-9 + 9j + 3)$$

$$-54 - 36j = (54B - 27C) + (-81B - 18C)j$$

Since the real part on the left must equal the real part on the right,

$$(1) \quad -54 = 54B - 27C$$

and the imaginary parts must also be equal

$$(2) \quad -36 = -81B - 18C$$

from (1)  $C = +2 + 2B$

substituting into (2)

$$-36 = -81B - 18(2 + 2B)$$

$$0 = -117B$$

$$\underline{B = 0}$$

$$C = 2 + 2B$$

$$\underline{C = 2}$$

$$\text{let } s = j : 2 - 7j - 27 + 51j + 27 = A(8)(2 + 3j) + (jB + C)$$

$$(j)(2 + 3j) + (jD + E)(j)(8)$$

$$2 + 44j = 16 + 24j + 4j - 6 - 8D + 8jE$$

$$-8 + 16j = -8D + (8E)j$$

Setting real and imaginary parts equal,

$$-8 = -8D$$

$$\underline{D = 1}$$

$$16 = 8E$$

$$\underline{E = 2}$$

$$\frac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{s^4 + 7s^3 + 27s^2 + 51s + 27} = \frac{1}{s} + \frac{2}{s^2 + 9} + \frac{s + 2}{s^2 + 3s + 3}$$

---


$$4) \quad \frac{7s^4 - 3s^3 + 56s^2 - 17s + 107}{(s-1)(s^2+4)^2} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+4)} + \frac{Ds+E}{(s^2+4)^2}$$

$$= \frac{A(s^2+4)^2 + (Bs+C)(s-1)(s^2+4) + (Ds+E)(s-1)}{(s-1)(s^2+4)^2}$$

Setting numerators equal:

$$7s^4 - 3s^3 + 56s^2 - 17s + 107 = A(s^2+4)^2 + (Bs+C)(s-1)(s^2+4) + (Ds+E)(s-1)$$

$$\text{Let } s = +1 : 7 - 3 + 56 = 17 + 107 = 25A$$

$$150 = 25A$$

$$\underline{A = 6}$$

Multiply it out

$$7s^4 - 3s^3 + 56s^2 - 17s + 107 = 6(s^4 + 8s^2 + 16) + (Bs^4 - Bs^3 + 4Bs^2 - 4Bs - Cs^2 + 4Cs - 4C) + (Ds^2 + Es - Ds - E)$$

Equating like coefficients,

$$s^4 : \quad 7 = 6 + B \Rightarrow B = 1$$

$$s^3 : \quad -3 = 0 - B + C \Rightarrow C = -2$$

$$s^2 : \quad 56 = 48 + 4B - C + D \Rightarrow D = 2$$

$$s^1 : \quad -17 = -4B + 4C + E - D \Rightarrow E - D = -5$$

$$\Rightarrow E = -3$$

B.

$$1) \quad \dot{y} + y = 5, \quad y(0) = 0$$

$$(s + 1) Y(s) = \frac{5}{s}$$

$$Y(s) = \frac{5}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1) + Bs}{s(s+1)}$$

$$\therefore 5 = A(s+1) + Bs$$

$$s = -1 : \quad 5 = -B$$

$$s = 0 : \quad 5 = A$$

$$Y(s) = \frac{5}{s} - \frac{5}{s+1}$$

$$y(t) = 5 - 5e^{-t}$$

$$2) \quad \dot{y} + y = e^{-3t}, \quad y(0) = 1$$

$$sY(s) - y(0) + Y(s) = \frac{1}{s+3}$$

$$Y(s) = \frac{1}{(s+1)(s+3)} + \frac{1}{s+1}$$

NOTE: We will use partial fractions on only part of the expression, since we have an inverse transform for  $\frac{1}{s+1}$ .

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \frac{A(s+3) + B(s+1)}{(s+1)(s+3)}$$

$$1 = A(s+3) + B(s+1)$$

$$s = -1 : 1 = 2A \longrightarrow A = 1/2$$

$$s = -3 : 1 = -2B \longrightarrow B = -1/2$$

$$Y(s) = \frac{1}{s+1} + \frac{1/2}{s+1} - \frac{1/2}{s+3}$$

$$Y(s) = \frac{3/2}{s+1} - \frac{1/2}{s+3}$$

$$y(t) = 3/2 e^{-t} - 1/2 e^{-3t}$$

$$3) \quad \dot{x} + 2x = \sin t, \quad x(0) = 5$$

$$sX(s) - x(0) + 2X(s) = \frac{1}{s^2 + 1}$$

$$X(s) = \frac{1}{(s^2 + 1)(s + 2)} + \frac{5}{s + 2}$$

Transform available so do not include in partial fraction expansion.

$$\frac{1}{(s^2 + 1)(s + 2)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 2} = \frac{(As + B)(s + 2) + C(s^2 + 1)}{(s^2 + 1)(s + 2)}$$

$$1 = (As + B)(s + 2) + C(s^2 + 1)$$

$$s = -2 : 1 = 5C \longrightarrow C = 1/5$$

$$s = 0 : 2B + C \longrightarrow B = 2/5$$

$$s = 1 : 3A + 3B + 2C \longrightarrow A = -1/5$$

$$X(s) = \frac{-1/5 s}{s^2 + 1} + \frac{2/5}{s^2 + 1} + \frac{5 \cdot 1/5}{s + 2}$$

$$x(t) = -1/5 \cos t + 2/5 \sin t + 5 \cdot 1/5 e^{-2t}$$

$$4) \quad x + 2\ddot{x} + 9x = 2 \sin 3t \quad x(0) = \dot{x}(0) = 0$$

$$(s^2 + 2s + 9) X(s) = \frac{6}{s^2 + 9}$$

$$X(s) = \frac{6}{(s^2 + 9)(s^2 + 2s + 9)} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 + 2s + 9}$$

$$6 = (As + B)(s^2 + 2s + 9) + (Cs + D)(s^2 + 9)$$

$$s = 0 : 6 = 9B + 9D$$

$$s = 3j : 6 = (3Aj + B)(-9 + 6j + 9) = (3Aj + B)(6j)$$

$$6 = -18A + 6Bj$$

$$A = -1/3$$

$$B = 0$$

$$D = 2/3$$

$$s = 1 : 6 = 12A + (C + D)10$$

$$6 = 12(-1/3) + 10C + 10(2/3)$$

$$C = 1/3$$

$$X(s) = \frac{-1/3 s}{s^2 + 9} + \frac{1/3 s + 2/3}{s^2 + 2s + 9}$$

$$X(s) = \frac{-1/3 s}{s^2 + 9} + \frac{1/3 (s + 2)}{(s + 1)^2 + 8}$$

$$= \frac{-1/3 s}{s^2 + 9} + \frac{1/3 (s + 1)}{(s + 1)^2 + 8} + \frac{1/3}{(s + 1)^2 + 8}$$

$$x(t) = -1/3 \cos 3t + 1/3 e^{-t} \cos 2\sqrt{2}t + 1/(6\sqrt{2}) e^{-t} \sin 2\sqrt{2}t.$$

$$5) \quad x + 5\dot{x} + 6x = 3e^{-3t}, \quad x(0) = \dot{x}(0) = 1$$

$$s^2 X(s) - sx(0) - \dot{x}(0) + 5sX(s) - 5x(0) + 6X(s) = \frac{3}{s + 3}$$

$$(s^2 + 5s + 6) X(s) = \frac{3}{s + 3} + s + 6 = \frac{s^2 + 9s + 21}{s + 3}$$

$$X(s) = \frac{s^2 + 9s + 21}{(s+3)(s^2 + 5s + 6)} = \frac{s^2 + 9s + 21}{(s+3)^2(s+2)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+2}$$

$$s^2 + 9s + 21 = A(s+3)(s+2) + B(s+2) + C(s+3)^2$$

$$s = -3 : 9 - 27 + 21 = -B \quad B = -3$$

$$s = -2 : 4 - 18 + 21 = C \quad C = 7$$

$$s = 0 : 21 = 6A + 2B + 9C \quad A = -6$$

$$x(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t}$$

$$6) \quad x + 2\dot{x} + 10x = 3t + \frac{3}{5}, \quad x(0) = 3, \quad \dot{x}(0) = -\frac{27}{10}$$

$$s^2 X(s) = sx(0) - \dot{x}(0) + 2sX(s) = 2x(0) + 10X(s) = \frac{3}{s^2} + \frac{3/5}{s}$$

$$(s^2 + 2s + 10) X(s) = \frac{3}{s^2} + \frac{3/5}{s} + sx(0) + 2x(0) + \dot{x}(0)$$

$$= \frac{3}{s^2} + \frac{3/5}{s} + 3s + 6 - \frac{27}{10}$$

$$= \frac{3s^3 + 33/10 s^2 + 6/10 s + 3}{s^2}$$

$$X(s) = \frac{3s^3 + 33/10 s^2 + 6/10 s + 3}{s^2(s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{(s^2 + 2s + 10)}$$

$$3s^3 + \frac{33}{10}s^2 + \frac{6}{10}s + 3 = As(s^2 + 2s + 10) + B(s^2 + 2s + 10) + (Cs + D)s^2$$

$$s = 0 : 3 = 10B \quad \underline{B = \frac{3}{10}}$$

$$s = 1 : 3 + \frac{33}{10} + \frac{6}{10} + 3 = A(13) + \frac{3}{10}(13) + (C + D)$$

$$\text{or } \underline{13A + C + D = 6}$$

$$s + j : -3j - \frac{33}{10} + \frac{6}{10}j + 3 = Aj(-1 + 2j + 10) + \frac{3}{10}(-1 + 2j + 10) + (Cj + D)(-1)$$

Setting imaginary and real parts equal

$$-\frac{24}{10} = 9A + \frac{6}{10} - C$$

$$\underline{C = 3 + 9A}$$

and

$$-\frac{3}{10} = -2A + \frac{27}{10} - D$$

$$D = -2A + 3$$

substituting into

$$13A + C + D = 6$$

$$13A = 3 + 9A - 2A + 3 = 6$$

$$\underline{A = 0}$$

then

$$\underline{C = 3}$$

and

$$\underline{D = 3}$$

$$X(s) = \frac{3/10}{s^2} + \frac{3s + 3}{s^2 + 2s + 10} = \frac{3/10}{s^2} + 3 \frac{s + 1}{(s + 1)^2 + 3^2}$$

$$\underline{x(t) = \frac{3}{10} t + 3e^{-t} \cos 3t}$$

■ 1.15 PROBLEM SET V

Solve the following problems using Laplace Transforms.

$$1. \quad \dot{x} + 3x - y = 1 \qquad x(0) = y(0) = 0$$

$$\dot{x} + 8x + \dot{y} = 2$$

$$2. \quad \dot{x} - 3x - 6y = e^{-2t} \qquad x(0) = y(0) = 0$$

$$\dot{y} + \dot{x} - 3y = 1$$

■ 1.16 SOLUTION TO PROBLEM SET V

$$1. \quad \dot{x} + 3x - y = 1$$

$$x(0) = y(0) = 0$$

$$\dot{x} + 8x + \dot{y} = 2$$

$$sX(s) - x(0) + 3X(s) - Y(s) = \frac{1}{s}$$

$$sX(s) - x(0) + 8X(s) + sY(s) - y(0) = \frac{2}{s}$$

$$(s + 3)X(s) - Y(s) = \frac{1}{s}$$

$$(s + 8)X(s) + sY(s) = \frac{2}{s}$$

$$\begin{vmatrix} s + 3 & -1 \\ s + 8 & s \end{vmatrix} X(s) = \begin{vmatrix} \frac{1}{s} & -1 \\ \frac{2}{s} & s \end{vmatrix} = 1 + \frac{2}{s} = \frac{s + 2}{s}$$

$$\begin{vmatrix} s + 3 & -1 \\ s + 8 & s \end{vmatrix} Y(s) = \begin{vmatrix} s + 3 & \frac{1}{s} \\ s + 8 & \frac{2}{s} \end{vmatrix} = \frac{2}{s}(s + 3) - \frac{1}{s}(s + 8) = \frac{s - 2}{s}$$

$$\begin{vmatrix} s + 3 & -1 \\ s + 8 & s \end{vmatrix} = s^2 + 3s + s + 8 = s^2 + 4s + 8$$

$$X(s) = \frac{s + 2}{s(s^2 + 4s + 8)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}$$

$$A(s^2 + 4s + 8) + s(Bs + C) = s + 2$$

$$s = 0: 8A = 2 \quad s^2: A + B = 0 \quad s: 4A + C = 1$$

$$A = 1/4$$

$$B = -1/4$$

$$C = 0$$

$$\therefore X(s) = \frac{1/4}{s} + \frac{-1/4 s}{(s+2)^2 + 4} = \frac{1/4}{s} - 1/4 \left[ \frac{s+2}{(s+2)^2 + 4} - \frac{2}{(s+2)^2 + 4} \right]$$

$$x(t) = 1/4 \left[ 1 - e^{-2t} (\cos 2t - \sin 2t) \right]$$

$$Y(s) = \frac{s-2}{s(s^2 + 4s + 8)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8}$$

$$A(s^2 + 4s + 8) + s(Bs + C) = s - 2$$

$$s = 0: 8A = -2 \quad s^2: A + B = 0 \quad s: 4A + C = 1$$

$$A = -1/4$$

$$B = 1/4$$

$$C = 2$$

$$Y(s) = 1/4 \left[ \frac{-1}{s} + \frac{s+8}{(s+2)^2 + 4} \right] = 1/4 \left[ \frac{-1}{s} + \frac{s+2}{(s+2)^2 + 4} + \frac{6}{(s+2)^2 + 4} \right]$$

$$y(t) = 1/4 \left[ -1 + e^{-2t} (\cos 2t + 3 \sin 2t) \right]$$

$$2. \quad \dot{x} - 3x - 6y = e^{-2t}$$

$$x(0) = y(0) = 0$$

$$\dot{y} + x - 3y = 1$$

$$(s - 3) X(s) - 6Y(s) = \frac{1}{s + 2}$$

$$sX(s) + (s - 3) Y(s) = \frac{1}{s}$$

$$\begin{vmatrix} s - 3 & -6 \\ s & s - 3 \end{vmatrix} X(s) = \begin{vmatrix} \frac{1}{s + 2} & -6 \\ \frac{1}{s} & s - 3 \end{vmatrix} = \frac{s - 3}{s + 2} + \frac{6}{s} = \frac{s^2 + 3s + 12}{s(s + 2)}$$

$$\begin{vmatrix} s - 3 & -6 \\ s & s - 3 \end{vmatrix} Y(s) = \begin{vmatrix} s - 3 & \frac{1}{s + 2} \\ s & \frac{1}{s} \end{vmatrix} = \frac{s - 3}{s} - \frac{s}{s + 2} = \frac{-s - 6}{s(s + 2)}$$

$$\begin{vmatrix} s - 3 & -6 \\ s & s - 3 \end{vmatrix} = s^2 - 6s + 9 + 6s = s^2 + 9$$

$$X(s) = \frac{s^2 + 3s + 12}{s(s + 2)(s^2 + 9)} = \frac{A}{s} + \frac{B}{s + 2} + \frac{Cs + D}{s^2 + 9}$$

$$A(s + 2)(s^2 + 9) + Bs(s^2 + 9) + (Cs + D)(s)(s + 2) = s^2 + 3s + 12$$

$$s = 0: 18A = 12$$

$$s = -2: -2B(4 + 9) = 4 - 6 + 12$$

$$\underline{A = 2/3}$$

$$-26B = 10$$

$$\underline{B = -5/13}$$

$$s^3: A + B + C = 0$$

$$s: 9A + 9B + 2D = 3$$

$$C = 5/13 - 2/3 = \frac{15 - 26}{39}; \quad 2D = 3 - 6 + \frac{45}{13} = \frac{45 - 39}{13} = \frac{6}{13}$$

$$\underline{C = -\frac{11}{39}}$$

$$\underline{D = \frac{3}{13}}$$

$$X(s) = \frac{2/3}{s} - \frac{5/13}{s+2} - \frac{11/39s}{s^2+9} + \frac{3/13}{s^2+9}$$

$$x(t) = 2/3 - 5/13e^{-2t} - 11/39 \cos 3t + 1/13 \sin 3t$$

$$Y(s) = \frac{-s-6}{s(s+2)(s^2+9)} = \frac{A}{s} + \frac{B}{s+2} + \frac{Cs+D}{s^2+9}$$

$$A(s+2)(s^2+9) + Bs(s^2+9) + (Cs+D)(s)(s+2) = -s-6$$

$$s=0: 18A = -6 \quad s=-2: -2B(13) = -4$$

$$A = -1/3$$

$$\underline{B = 2/13}$$

$$s = j3: (j3C + D)(j3)(s + j3) = -j3 - 6$$

$$j3(j3C + 2D + j3D - 9C) = -j3 - 6$$

$$-18C - 9D - j27C + j6D = -6 - j3$$

$$+18C + 9D = +6$$

$$6D - 27C = -3$$

$$6C + 3D = 2$$

$$27C - 6D = 3$$

$$12C + 6D = 4$$

$$\underline{27C - 6D = 3}$$

$$39C = 7$$

$$3D = 2 - \frac{42}{39} = \frac{78 - 42}{39} = \frac{36}{39} = \frac{12}{13}$$

$$\underline{C = \frac{7}{39}}$$

$$\underline{D = \frac{4}{13}}$$

$$Y(s) = \frac{-1/3}{s} + \frac{2/13}{s+2} + \frac{7/39s}{s^2+9} + \frac{4/13}{s^2+9}$$

$$y(t) = -1/3 + 2/13e^{-2t} + 7/39 \cos 3t + 4/39 \sin 3t$$

# CHAPTER EQUATIONS OF MOTION

# 2

(Revised November 1973)

## 2.1 INTRODUCTION

These notes are written as a general classroom text for the theoretical approach to Stability and Control in the course curriculum of the USAF Test Pilot School.

The theoretical discussion will, of necessity, incorporate certain simplifying assumptions. These simplifying assumptions are made in order to make the main elements of the subject more clear. The equations developed are by no means suitable for design of modern aircraft, but the basic method of attacking the problem is valid. Now that analog and digital computers are available, the aircraft designers' more rigorous theoretical calculations, modified by data obtained from the wind tunnel, often give results which closely predict the flying qualities of new airplanes. However, neither the theoretical nor the wind tunnel results are infallible. Therefore, there is still a valid requirement for the test pilot in the development cycle of new aircraft.

## 2.2 TERMS AND SYMBOLS

There will be many terms and symbols used during the stability and control phase. Some of these will be familiar, but many will be new. It will be a great asset to be able to recall at a glance the definitions represented by these symbols. Below is a condensed list of the terms and symbols used in this course:

### 2.2.1 Terms

Stability Derivatives - Nondimensional quantities expressing the variation of the force or moment coefficient with a disturbance from steady flight.

$$C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha} \quad (2.1)$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta} \quad (2.2)$$

Stability Parameters - A quantity that expresses the variation of force or moment on aircraft caused by flight or by a disturbance from steady flight.

$$M_u = \frac{\rho U S c}{I_y} \left[ C_m + \frac{U}{2} \frac{\partial C_m}{\partial u} \right] \quad \text{(Change in pitching moment caused by a change in velocity)} \quad (2.3)$$

$$L_q = \frac{\rho S c}{4m} C_{L_q} \quad \text{(Change in lift caused by a change in pitch rate)} \quad (2.4)$$

Static Stability - The initial tendency of an airplane to return to steady state flight after a disturbance.

Dynamic Stability - The time history of the response of an airplane to a disturbance, in which the aircraft ultimately returns to a steady state flight.

Neutral Stability -

- a) Static - The airplane would have no tendency to move from its disturbed condition.
- b) Dynamic - The airplane would sustain a steady oscillation caused by a disturbance.

Static Instability - A characteristic of an aircraft such that when disturbed from steady flight, its tendency is to depart further or diverge from the original condition of steady flight.

Dynamic Instability - Time history of an aircraft response to a disturbance in which the aircraft ultimately diverges to departure or destruction.

Flight Control Sign Convention - Any control movement or deflection that causes a positive movement or moment on the airplane shall be considered a positive control movement. This sign convention does not conform to the convention used by NASA and some reference text books. This convention is the easiest to remember and is used at the Flight Test Center, therefore, it will be used in the School.

Degrees of Freedom - The number of paths that a physical system is free to follow.



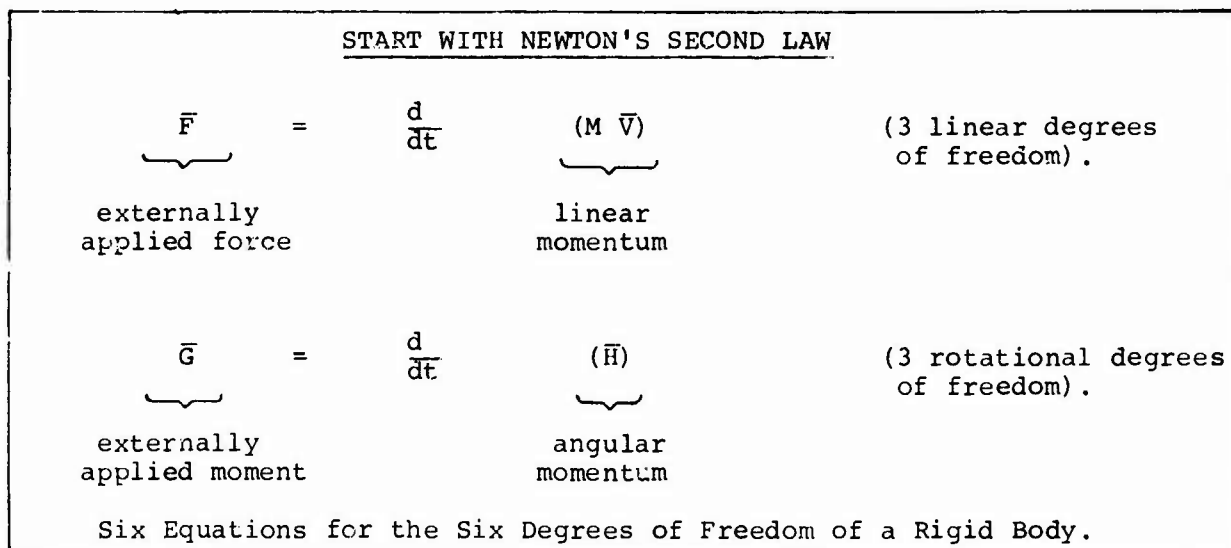
## 2.2.2 SYMBOLS

<u>Symbol</u>	<u>Definition</u>
a.c.	Aerodynamic Center: A point located on the wing chord (approximately one quarter of the chord length back of the leading edge for subsonic flight) about which the moment coefficient is practically constant for all angles of attack.
C	Chordwise Force: The component of the resultant aerodynamic force that is parallel to the aircraft reference axis, (i.e., fuselage reference line).
c	Mean Aerodynamic Chord: The theoretical chord for a wing which has the same force vector as the actual wing (also MAC).
c.p.	Center of Pressure: Theoretical point on the chord through which the resultant force acts.
D	Drag: The component of the resultant aerodynamic force parallel to the relative wind. It too must be specified whether this applies to a complete aircraft or to parts thereof.
$\bar{F}$	Applied force vector.
$F_a, F_e, F_r$	Control forces on the aileron, elevator, and rudder, respectively.
$F_x, F_y, F_z$	Components of applied forces on respective body axes.
$\bar{G}$	Applied moment vector.
$G_x, G_y, G_z$	Components of applied moments on respective body axes.
$\bar{H}$	Angular momentum vector.
HM	Hinge Moment: A moment which tends to restore or move a control surface to or from a condition to equilibrium.
$H_x, H_y, H_z$	Components of the angular momentum vector on the body axes.
I	Moment of Inertia: With respect to any given axis, the moment of inertia is the sum of the products of the mass of each elementary particle by the square of its distance from the axis. It is a measure of the resistance of a body to angular acceleration.
$\bar{i}, \bar{j}, \bar{k}$	Unit vectors in the body axis system.

<u>Symbol</u>	<u>Definition</u>
$I_x, I_y, I_z$	Moments of inertia about respective body axes.
$I_{xy}, I_{yz}, I_{xz}$	Products of inertia.
L	Lift: The component of the resultant aerodynamic force perpendicular to the relative wind. It must be specified whether this applies to a complete aircraft or to parts thereof.
$L, M, N$	Aerodynamic moments about x, y, and z body axes.
N	Normal Force: The component of the resultant aerodynamic force that is perpendicular to the aircraft reference axis.
$p, q, r$	Angular rates about the x, y, and z body axes, respectively.
R	Resultant Aerodynamic Force: The vector sum of the lift and drag forces on an airfoil or airplane.
S	Wing area.
$U_0$	Component of velocity along the x body axis at zero time (i.e., initial condition).
$u, v, w$	Components of velocity along the x, y, and z body axes.
$X, Y, Z$	Aerodynamic force components on respective body axes (Caution: Also used as axes in "Moving Earth Axis System" in derivation of Euler angle equation. Differentiation should be obvious).
$x, y, z$	Axes in the body axis system.
$\alpha$	Angle of attack.
$\beta$	Sideslip angle.
$\delta_a, \delta_e, \delta_r$	Deflection angle of the ailerons, elevator, and rudder, respectively.
$\theta, \phi, \psi$	Euler angles: pitch, roll, and yaw, respectively.
$\bar{\omega}$	Total angular velocity vector of an aircraft.

## 2.3 OVERVIEW

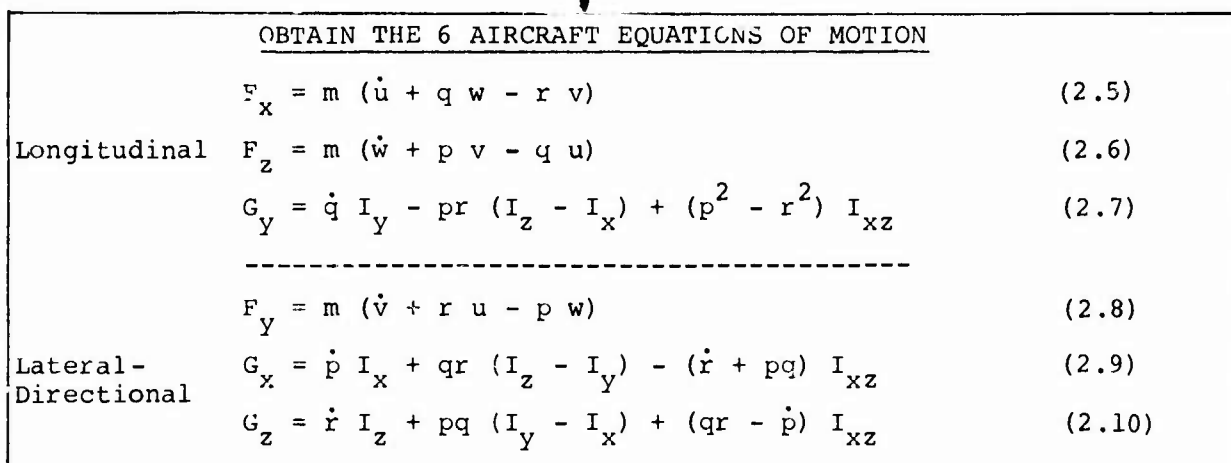
The purpose of this section is to derive a set of equations that describes the motion of an airplane. An airplane has 6 degrees of freedom (i.e., it can move forward, sideways and down and it can rotate about its axes with yaw, pitch, and roll). In order to solve for these 6 unknowns, 6 simultaneous equations will be required. To derive these the following relations will be used.



↓

Equations are valid with  
 respect to inertial space only.

↓



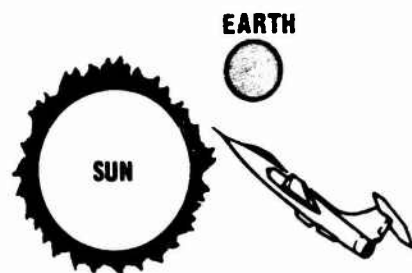
The Left-Hand Side (LHS) of the equation represents the forces and moments on the airplane while the Right-Hand Side (RHS) stands for the airplane's response to these forces and moments. Before launching into the development of these equations it will be necessary first to cover some basics.

## 2.4 BASICS

### 2.4.1 Coordinate Systems

There are many coordinate systems that are useful in the analysis of vehicle motion. In accordance with general practice, all coordinate systems will be right-hand and orthogonal.

#### True Inertial Coordinate System



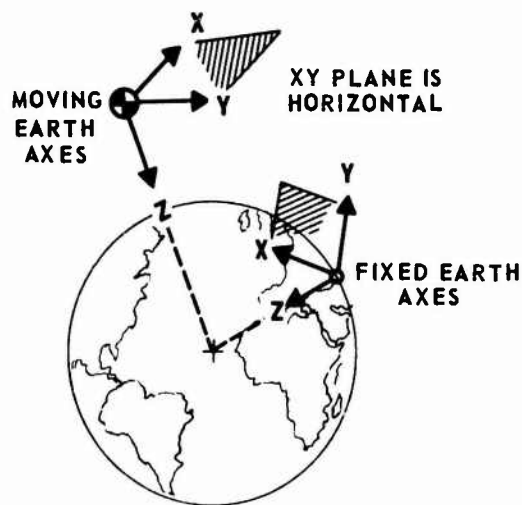
Location of origin: unknown

Approximation for space dynamics: the center of the sun.

Approximation for aircraft: the center of the earth.

Figure 2.1

#### The Earth Axis Systems



##### Location of Origin

Fixed System; arbitrary location  
Moving System; at the vehicle cg

The Z axis points toward center of the earth.

The XY Plane parallel to local horizontal.

The Orientation of the X axis is arbitrary; may be North or on the initial vehicle heading.

**NOTE:** There are two earth axis systems, the fixed and the moving. An example of a moving earth axis system is an inertial navigation platform. An example of a fixed earth axis is a radar site.

Figure 2.2

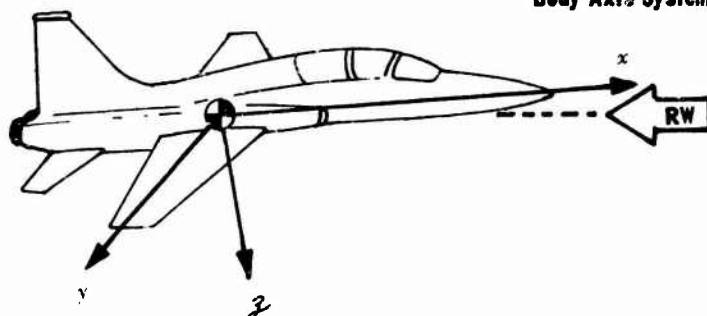
## Moving Earth Axis Systems

These coordinate systems are fixed to the vehicle. There are many different types, e.g.,

Body Axis System.  
Stability Axis System.  
Principal Axis System.  
Wind Axis System.

The body and the stability axis systems are the only two that will be used during this course.

### Body Axis System



The Unit Vectors are  $\bar{i}$   $\bar{j}$   $\bar{k}$ .

The Origin is at the cg.

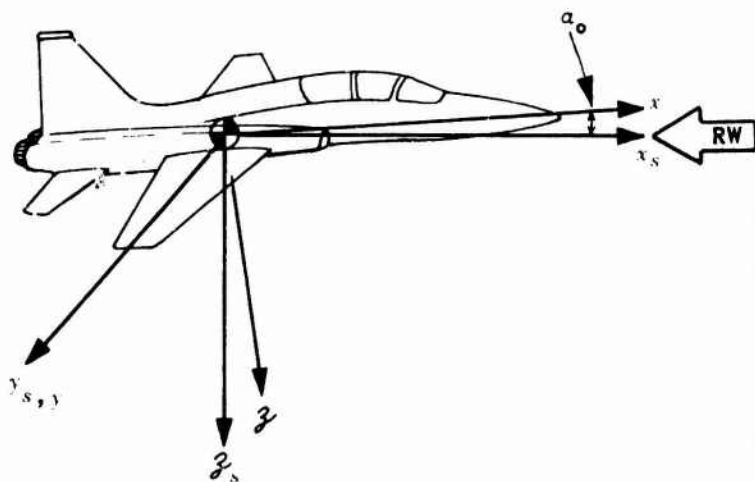
The  $xz$  plane is in the vehicle plane of symmetry.

The positive  $x$  axis points forward along the vehicle horizontal reference line.

The positive  $z$  axis points downward toward the bottom of the vehicle.

Figure 2.3

### Stability Axis System



The unit vectors are  $\bar{i}_s$ ,  $\bar{j}_s$ ,  $\bar{k}_s$ .

The origin is at the cg.

The positive  $x$  axis points forward coincident with the initial position of the relative wind.

The  $xz$  plane must remain in the vehicle plane of symmetry, hence this stability axis system is restricted to symmetrical initial flight conditions.

The positive  $z$  axis points downward toward the bottom of the vehicle.

$y_{\text{BODY}} = y_{\text{STAB}}$   
i.e., THE STABILITY  $xz$  PLANE  
REMAINS IN THE VEHICLE PLANE  
OF SYMMETRY

Figure 2.4

## 2.4.2 Vector Definitions

The Equations of Motion describe the vehicle motion in terms of four vectors. The components of these vectors resolved along the body axis system are shown below.

$\bar{F}$  - Total Linear Force (Applied)

$$\bar{F} = F_x \bar{i} + F_y \bar{j} + F_z \bar{k}$$

$\bar{G}$  - Total Moment (Applied)

$$\bar{G} = G_x \bar{i} + G_y \bar{j} + G_z \bar{k}$$

$$\bar{G} = \bar{G}_{\text{aerodynamic}} + \bar{G}_{\text{other sources}}$$

$$\bar{G}_{\text{aerodynamic}} = L_{\text{aero}} \bar{i} + M_{\text{aero}} \bar{j} + N_{\text{aero}} \bar{k}$$

$$\bar{G}_{\text{aerodynamic}} = \mathcal{L} \bar{i} + \mathcal{M} \bar{j} + \mathcal{N} \bar{k}$$

NOTE: Control deflections that tend to produce positive  $\mathcal{L}$ ,  $\mathcal{M}$ , or  $\mathcal{N}$ , are defined at USAF TPS to be positive.

$\bar{V}_T$  - True Velocity

$$\bar{V}_T = u \bar{i} + v \bar{j} + w \bar{k}$$

where

$u$  = forward velocity

$v$  = side velocity

$w$  = vertical velocity

$\alpha$  - Angle of Attack

For small  $\alpha$  and  $\beta$

$$V_T \cos \beta \approx V_T$$

$$\therefore \alpha \approx \sin^{-1} \frac{w}{V_T}$$

or

$$\alpha \approx \frac{w}{V_T}$$

$\beta$  - Sideslip Angle

$$\beta = \sin^{-1} \left( \frac{v}{V_T} \right)$$

For small  $\beta$

$$\beta \approx \frac{v}{V_T}$$

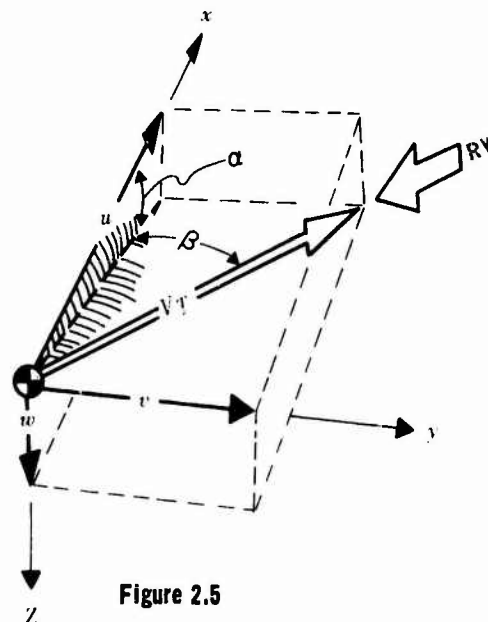


Figure 2.5

**CAUTION - OTHER DEFINITIONS ARE POSSIBLE**

$\bar{\omega}$  - Angular Velocity

$$\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k}$$

where

$p$  = roll rate

$q$  = pitch rate

$r$  = yaw rate

#### 2.4.3 Euler Angles - Transformation from the Moving Earth Axis System to the Body Axis System

There are several reasons for using Euler angles in this development. Some of them are:

- 1) Effect of aircraft weight is related to the body axes through Euler angles.
- 2) When an inertial navigation system (INS) is available, data can be taken directly in Euler angles.  $p$ ,  $q$  and  $r$  can then be determined through a transformation.

Euler angles are expressed in terms of YAW ( $\psi$ ), PITCH ( $\theta$ ) and ROLL ( $\phi$ ). The sequence (YAW, PITCH, ROLL) must be maintained to arrive at the proper set of Euler angles.

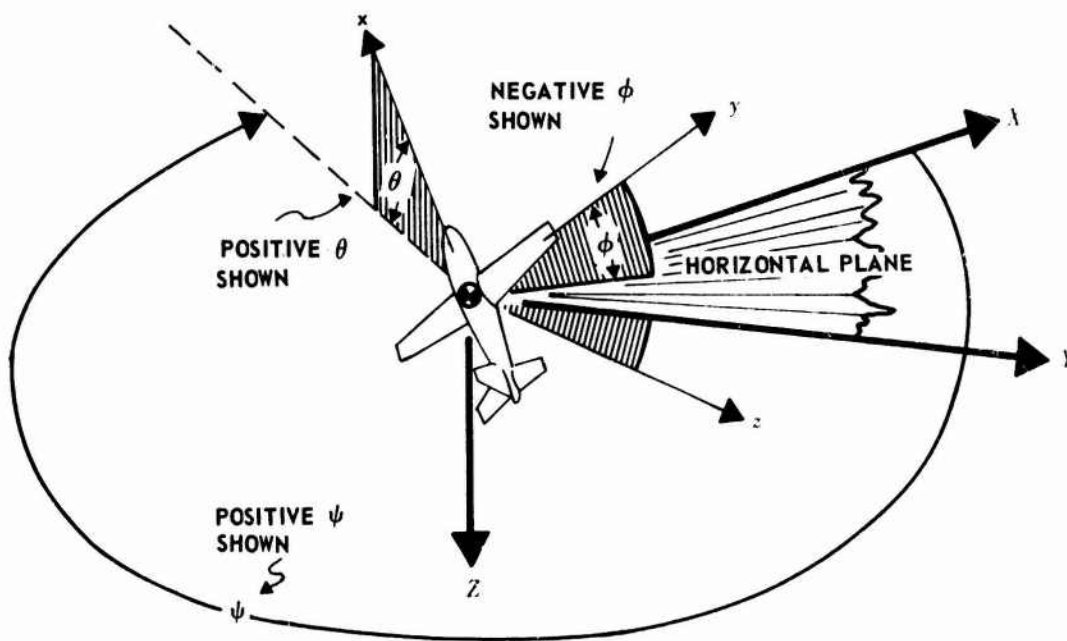


Figure 2.6

$\psi$  - Yaw Angle - The angle between the projection of  $x$  body axes onto the horizontal plane and the initial reference position of the  $X$  earth axis. (Yaw angle is the vehicle heading only if the initial reference is North).

$\theta$  - Pitch Angle - The angle measured in a vertical plane between the  $x$  body axis and the horizontal plane.

$\phi$  - Bank Angle - The angle, measured in the  $yz$  plane of the body system, between the  $y$  body axis and the horizontal plane.

Angular Velocity Transformation - The following relationships, derived by vector resolution, will be useful later in the study of dynamics.

$$p = \dot{\phi} - \dot{\psi} \sin \theta \quad (2.11)$$

$$q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta \quad (2.12)$$

$$r = \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi \quad (2.13)$$

The above equations transform the angular rates in the moving earth axis system ( $\dot{\psi}, \dot{\theta}, \dot{\phi}$ ) into angular rates about the body axis system ( $p, q, r$ ) for any aircraft attitude. For example, it is easy to see that when an aircraft is pitched up and banked, the vector  $\dot{\psi}$  will have components along the  $x, y$  and  $z$  body axis (figure 2.7). Remember,  $\dot{\psi}$  is the angular velocity about the  $Z$  axis of the Moving Earth Axis System (it can be thought of as the rate of change of aircraft heading). Although it is not shown in figure 2.7, the aircraft may have a value of  $\dot{\theta}$  and  $\dot{\phi}$ . In order to derive the transformation equations it is easier to analyze one vector at a time. First resolve the components of  $\dot{\psi}$  on the body axes. Then do the same with  $\dot{\theta}$  and  $\dot{\phi}$ . The components can then be added and the total transformation will result.

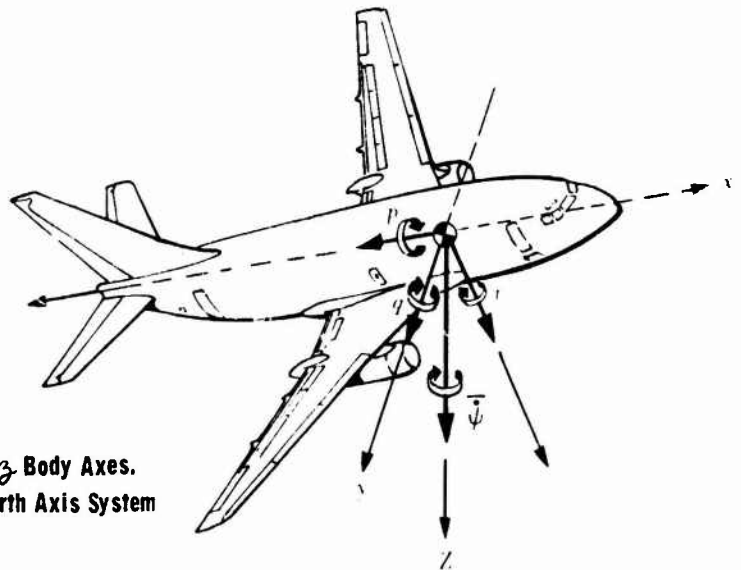


Figure 2.7 Components of  $\dot{\psi}$  Along  $x, y$ , and  $z$  Body Axes. The  $X$  and  $Y$  Axes of the Moving Earth Axis System Are Not Shown.

### Derivation of the Angular Velocity Transformation

Step 1 - Resolve the components of  $\dot{\psi}$  along the body axes for any aircraft attitude.

It is easy to see how  $\dot{\psi}$  reflects to the body axis by starting with an aircraft in straight and level flight and changing the aircraft attitude one angle at a time. In keeping with convention, the sequence of change will be yaw, pitch and bank.

First, it can be seen that the Z axis of the Moving Earth Axis System remains aligned with z axis of the body axis system regardless of the angle  $\psi$ ; therefore, the effect of  $\dot{\psi}$  on p, q, and r does not change with yaw angle,

$$\therefore r = \dot{\psi}$$

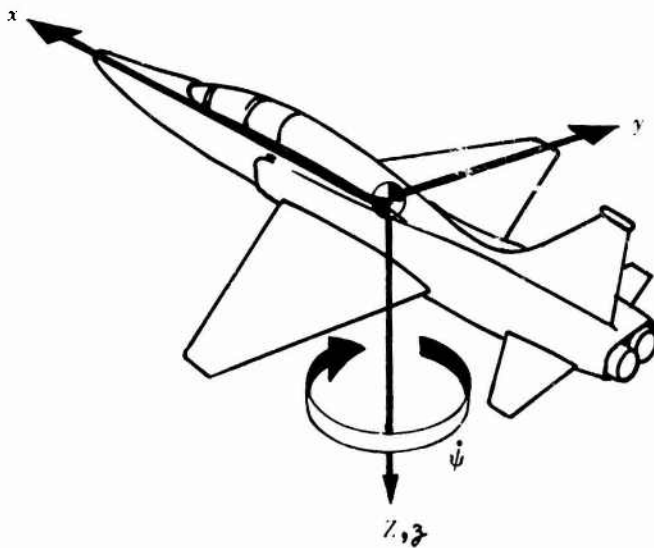


Figure 2.8

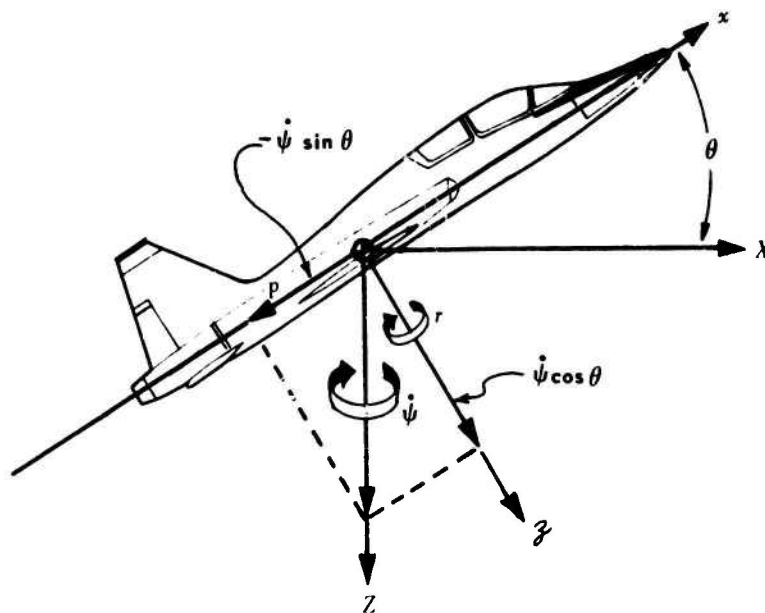


Figure 2.9

Next, consider pitch up, in this attitude,  $\dot{\psi}$  has components on the  $x$  and  $z$  body axes.

$$p = -\dot{\psi} \sin \theta$$

$$r = \dot{\psi} \cos \theta$$

The  $z$  axis is still perpendicular to the  $y$  body axis, so  $q$  is not effected by  $\dot{\psi}$  in this attitude.

Finally, bank the aircraft, leaving the pitch as it is.

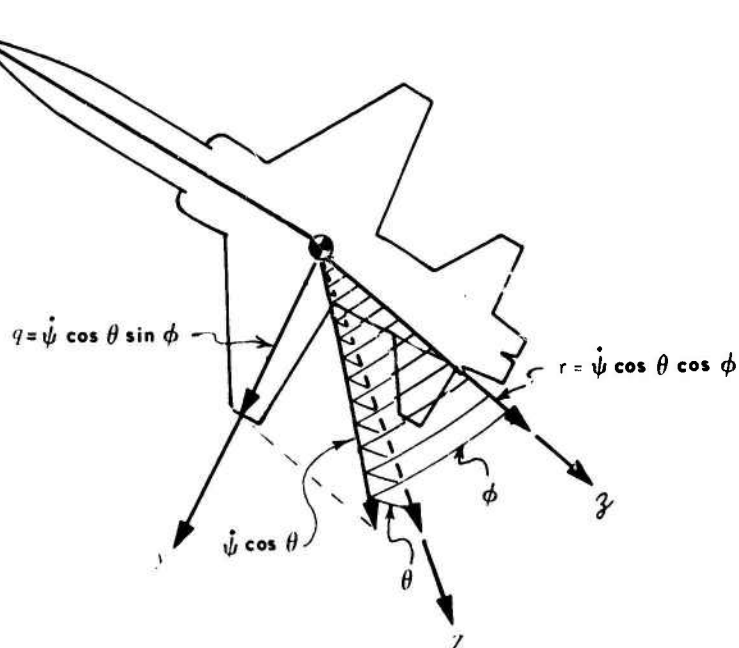


Figure 2.10

All of the components are now illustrated. Notice that roll did not change the effect of  $\dot{\psi}$  on  $p$ . The components, therefore, of  $\dot{\psi}$  in the body axes for any aircraft attitude are:

$$p = -\dot{\psi} \sin \theta$$

Effect of  $\dot{\psi}$  only.

$$q = \dot{\psi} \cos \theta \cos \phi$$

$$r = \dot{\psi} \cos \theta \sin \phi$$

Step 2 - Resolve the components of  $\dot{\theta}$  along the body axes for any aircraft attitude.

Remember  $\theta$  is the angle between the  $x$  body axis and the local horizontal. Once again, change the aircraft attitude by steps in the sequence of yaw, pitch and bank and analyze the effects of  $\dot{\theta}$ .

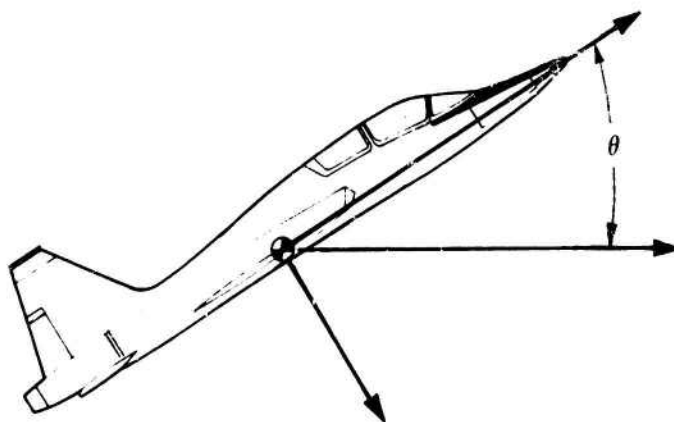


Figure 2.11

It can be seen immediately that the yaw angle has no effect. Likewise when pitched up, the  $y$  body axis remains in the horizontal plane. Therefore,  $\dot{\theta}$  is the same as  $q$  in this attitude.

$$q = \dot{\theta}$$

Now bank the aircraft.

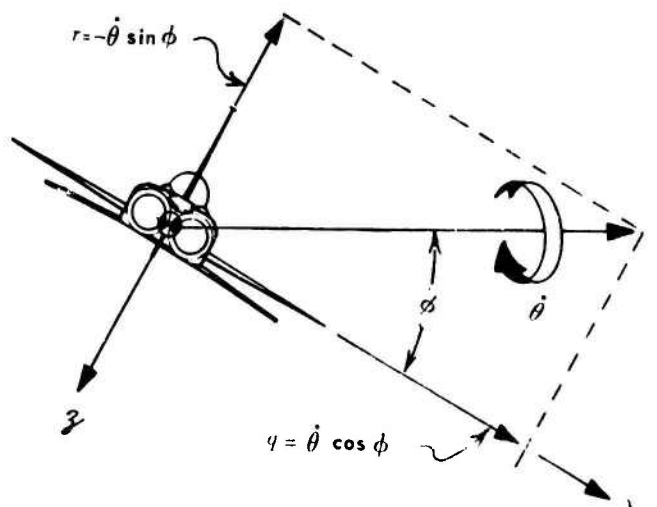


Figure 2.12

It can be seen from figure 2.12 that the components of  $\dot{\theta}$  on the body axes are

$$q = \dot{\theta} \cos \phi$$

$$r = -\dot{\theta} \sin \phi$$

Notice that  $p$  is not affected by  $\dot{\theta}$  since by definition  $\dot{\phi}$  is measured on an axis perpendicular to the  $x$  body axis.

Step 3 - Resolve the components of  $\dot{\phi}$  along the body axes.

This one is easy since by definition  $\dot{\phi}$  is measured along the  $x$  body axis. Therefore,  $\dot{\phi}$  affects the value of  $p$  only, or

$$p = \dot{\phi}$$

The components of  $\dot{\psi}$ ,  $\dot{\theta}$ , and  $\dot{\phi}$  along the  $x$ ,  $y$ , and  $z$  body axes for any aircraft attitude have been derived. These can now be summed to give the transformation equations.

$$p = \dot{\phi} - \dot{\psi} \sin \theta$$

$$q = \dot{\theta} \cos \phi + \dot{\psi} \sin \phi \cos \theta$$

$$r = \dot{\psi} \cos \phi \cos \theta - \dot{\theta} \sin \phi$$

#### 2.4.4 Assumptions

The following assumptions will be made to simplify the derivation of the equations of motion. The reasons for these assumptions will become obvious as the equations are derived.

Rigid Body - Aeroelastic effects must be considered separately.

Earth and Atmosphere are Assumed Fixed - Allows use of Moving Earth Axis System as an "inertial reference".

Constant Mass - Most motion of interest in stability and control takes place in a relatively short time.

The  $xz$  Plane is a Plane of Symmetry - This restriction is made to simplify the RHS of the equation. It allows the cancellation of certain terms containing products of inertia. The restriction can easily be removed by including these terms.

#### 2.5 RIGHT-HAND SIDE OF EQUATION

The RHS of the equation represents the aircraft response to any forces or moments that are applied to it. Through the application of Newton's Second Law, two vector relations can be used to derive the six required equations. These are the linear force and moment relations.

### 2.5.1 Linear Force Relation

The vector equation for response to an applied linear force is

$$\bar{F} = \frac{d(m\bar{V})}{dt} \quad (2.14)$$

or the change in momentum of an object is equal to the force applied to it.

This applies, of course, only with respect to inertial space. Therefore, the motion of a body is determined by all the forces applied to it including gravitational attraction of the earth, moon, sun, and even the stars. In most cases, the practical person disregards the effect of the moon, sun, and stars since their influence is extremely small. When considering the forces on an aircraft, the motion of the earth and atmosphere can also be disregarded since the forces resulting from the earth's rotation and coriolis effects are negligible when compared with the large aerodynamic and gravitational forces involved. This simplifies the derivation considerably. The equations can now be derived using either a fixed or moving earth axis system. For graphical clarity consider a fixed earth axis system. The vehicle represented by the dot has a total velocity vector that is changing in both magnitude and direction.

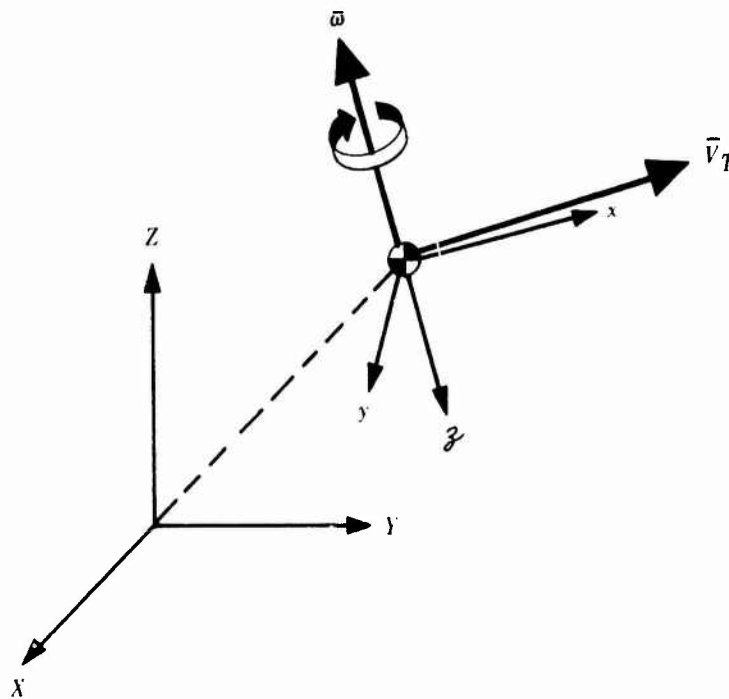


Figure 2.13

From vector analysis,

$$\left. \frac{d\bar{V}_T}{dt} \right|_{XYZ} = \left. \frac{d\bar{V}_T}{dt} \right|_{xyz} + \bar{\omega} \times \bar{V}_T$$

Substituting this into equation 2.14, and assuming mass is constant, the applied force is,

$$\bar{F} = m \left[ \left. \frac{d\bar{V}_T}{dt} \right|_{xyz} + \bar{\omega} \times \bar{V}_T \right]$$

which in component form is

$$\bar{F} = m \left[ \dot{u}\bar{i} + \dot{v}\bar{j} + \dot{w}\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p & q & r \\ u & v & w \end{vmatrix} \right]$$

Expanding

$$\bar{F} = m [\dot{u}\bar{i} + \dot{v}\bar{j} + \dot{w}\bar{k} + (qw - rv)\bar{i} - (pw - ru)\bar{j} + (pv - qu)\bar{k}]$$

Rearranging

$$\bar{F} = m [(\dot{u} + qw - rv)\bar{i} + (\dot{v} + ru - pw)\bar{j} + (\dot{w} + pv - qu)\bar{k}]$$

Now since

$$\bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$$

These three component equations result:

$$F_x = m (\dot{u} + qw - rv) \quad (2.15)$$

$$F_y = m (\dot{v} + ru - pw) \quad (2.16)$$

$$F_z = m (\dot{w} + pv - qu) \quad (2.17)$$

### 2.5.2 Moment Equations

Once again from Newton's Second Law,

$$\bar{G} = \frac{d(\bar{H})}{dt} \quad (2.18)$$

or the change in angular momentum is equal to the total applied moment.

#### Angular Momentum

Angular momentum should not be as difficult to understand as some people would like to make it. It can be thought of as linear momentum with a moment arm included.

Consider a ball swinging on the end of a string, at any instant of time,

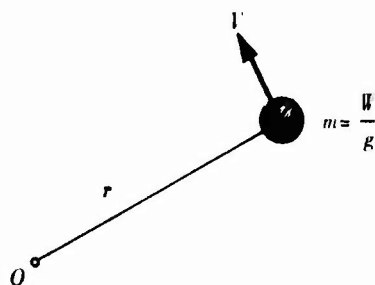


Figure 2.14

$$\text{Linear momentum} = m\bar{V}$$

and

$$\text{Angular momentum} = m\bar{r}\bar{V} \text{ (axis of rotation must be specified).}$$

Therefore, they are related in the same manner that forces relate to moments.

$$\text{Moment} = \text{Force} \cdot r$$

$$\text{Angular Momentum} = \text{Linear Momentum} \cdot r$$

and just as a force changes linear momentum, a moment will change angular momentum.

#### Angular Momentum of an Aircraft

Consider a small element of mass  $m_1$ , somewhere in the aircraft, a distance  $\bar{r}_1$  from the cg

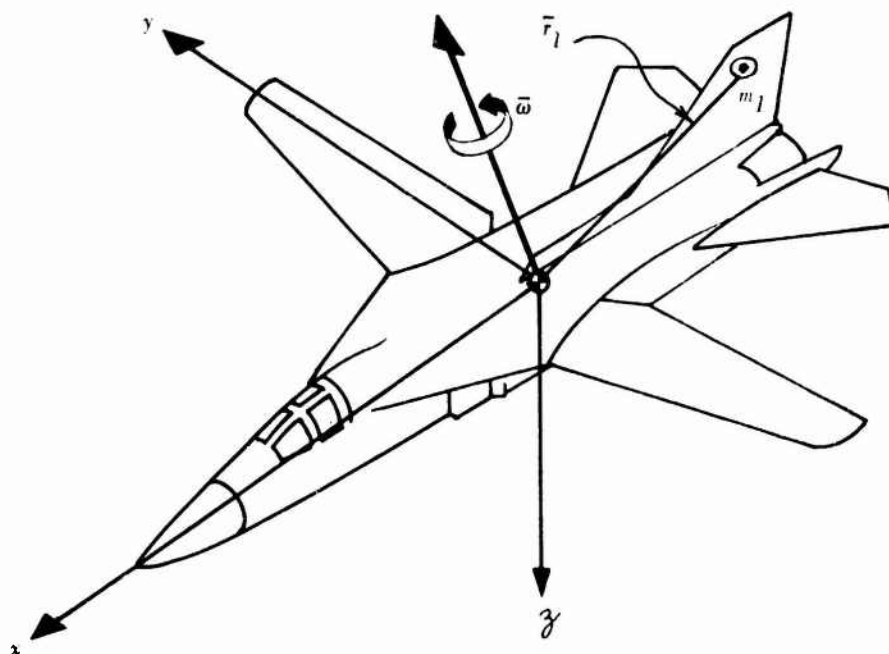


Figure 2.15

The airplane is rotating about all three axes so that

$$\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k} \quad (2.19)$$

and

$$\bar{r}_1 = x_1\bar{i} + y_1\bar{j} + z_1\bar{k} \quad (2.20)$$

The angular momentum of  $m_1$  is

$$\bar{H}_{m_1} = m_1 (\bar{r}_1 \times \bar{v}_1) \quad (2.21)$$

and

$$\bar{v}_1 = \left. \frac{d\bar{r}_1}{dt} \right|_{XYZ} \quad (\text{i.e., in the inertial frame})$$

From vector analysis

$$\left. \frac{d\bar{r}_1}{dt} \right|_{XYZ} = \left. \frac{d\bar{r}_1}{dt} \right|_{xyz} + \bar{\omega} \times \bar{r}_1 \quad (2.22)$$

Since the airplane is a rigid body  $\bar{r}_1$  does not change. Therefore the first term can be excluded, and the inertial velocity of the element  $m_1$  is

$$\bar{v}_1 = \bar{\omega} \times \bar{r}_1 \quad (2.23)$$

Substituting this into equation 2.21

$$\bar{H}_{m_1} = m_1 (\bar{r}_1 \times \bar{\omega} \times \bar{r}_1) \quad (2.24)$$

This is the angular momentum of the small element of mass  $m_1$ . In order to find the angular momentum of the whole airplane, take the sum of all the elements. Using notation in which the  $i$  subscript indicates any particular element and  $n$  is the total number of elements in the airplane,

$$\bar{H} = \sum_{i=1}^n m_i [\bar{r}_i \times \bar{\omega} \times \bar{r}_i] \quad (2.25)$$

where

$$m_i = \text{scalar}$$

$$\bar{r}_i = x_i\bar{i} + y_i\bar{j} + z_i\bar{k} \quad (2.26)$$

then

$$\bar{\omega} \times \bar{r}_i = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p & q & r \\ x_i & y_i & z_i \end{vmatrix} \quad (2.27)$$

In an effort to reduce the clutter, the subscripts will be left off. The determinant can be expanded to give,

$$\vec{\omega} \times \vec{r} = (qz - ry)\vec{i} + (rx - pz)\vec{j} + (py - qx)\vec{k} \quad (2.28)$$

therefore, equation 2.25 becomes

$$\vec{H} = \sum m \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ (qz-ry) & (rx-pz) & (py-qx) \end{vmatrix} \quad (2.29)$$

So the components of  $\vec{H}$  are

$$H_x = \sum m y (py - qx) - \sum m z (rx - pz) \quad (2.30)$$

$$H_y = \sum m z (qz - ry) - \sum m x (py - qx) \quad (2.31)$$

$$H_z = \sum m x (rx - pz) - \sum m y (qz - ry) \quad (2.32)$$

Rearranging the equations

$$H_x = p \sum m (y^2 + z^2) - q \sum m xy - r \sum m xz \quad (2.33)$$

$$H_y = q \sum m (z^2 + x^2) - r \sum m yz - p \sum m xy \quad (2.34)$$

$$H_z = r \sum m (x^2 + y^2) - p \sum m xz - q \sum m yz \quad (2.35)$$

Define moments of inertia

$$I_x = \sum m (y^2 + z^2)$$

$$I_y = \sum m (x^2 + z^2)$$

$$I_z = \sum m (x^2 + y^2)$$

These are a measure of resistance to rotation - they are never zero.

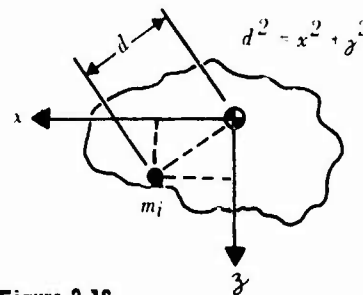


Figure 2.16

Define products of inertia

$$I_{xy} = \sum mxy$$

$$I_{yz} = \sum myz$$

$$I_{xz} = \sum mxz$$

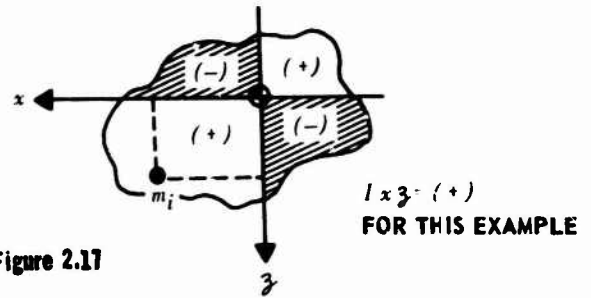


Figure 2.17

These are a measure of symmetry. They are zero for views having a line of symmetry.

The angular momentum of a rigid body is therefore:

$$\bar{H} = H_x \bar{i} + H_y \bar{j} + H_z \bar{k} \quad (2.36)$$

So that

$$H_x = pI_x - qI_{xy} - rI_{xz} \quad (2.37)$$

$$H_y = qI_y - rI_{yz} - pI_{xy} \quad (2.38)$$

$$H_z = rI_z - pI_{xz} - qI_{yz} \quad (2.39)$$

#### Simplification of Angular Moment Equation for Symmetric Aircraft

A symmetric aircraft has two views that contain a line of symmetry and hence two products of inertia that are zero. The angular momentum of a symmetric aircraft therefore simplifies to:

$$\bar{H} = (pI_x - rI_{xz})\bar{i} + qI_y\bar{j} + (rI_z - pI_{xz})\bar{k} \quad (2.40)$$

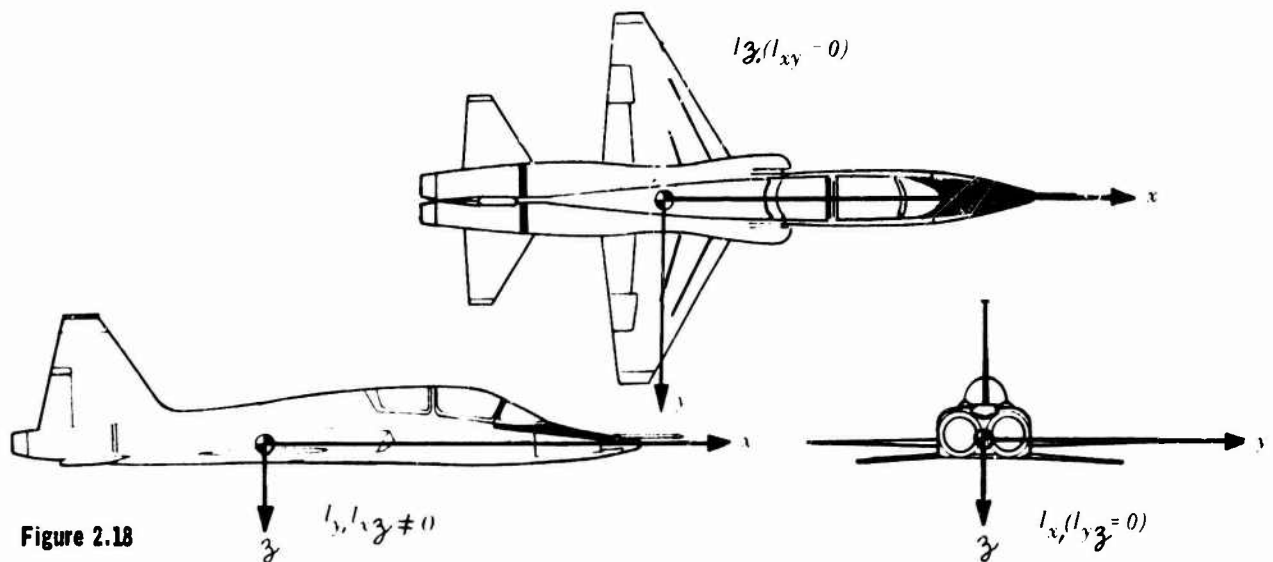


Figure 2.18

### Derivation of the Three Rotational Equations

The equation for angular momentum can now be substituted into the moment equation. Remember

$$\bar{G} = \left. \frac{d\bar{H}}{dt} \right|_{XYZ} \quad (2.41)$$

This is in the inertial frame. Expressed in the fixed body axis system, the equation becomes:

$$\bar{G} = \left. \frac{d\bar{H}}{dt} \right|_{xyz} + \bar{\omega} \times \bar{H} \quad (2.42)$$

which is

$$\bar{G} = \dot{H}_x \bar{i} + \dot{H}_y \bar{j} + \dot{H}_z \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p & q & r \\ H_x & H_y & H_z \end{vmatrix} \quad (2.43)$$

Remember

$$\bar{H} = (pI_x - rI_{xz})\bar{i} + qI_y\bar{j} + (rI_z - pI_{xz})\bar{k} \quad (2.44)$$

Since the body axis system is used, the moments of inertia and the products of inertia are constant. Therefore, by differentiating and substituting, the moment equation becomes

$$\bar{G} = (\dot{p}I_x - \dot{r}I_{xz})\bar{i} + \dot{q}I_y\bar{j} + (\dot{r}I_z - \dot{p}I_{xz})\bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p & q & r \\ (pI_x - rI_{xz}) & qI_y & (rI_z - pI_{xz}) \end{vmatrix} \quad (2.45)$$

Therefore, the component equations are,

$$G_x = \dot{p}I_x + qr(I_z - I_y) - (\dot{r} + pq)I_{xz} \quad (2.46)$$

$$G_y = \dot{q}I_y - pr(I_z - I_x) + (p^2 - r^2)I_{xz} \quad (2.47)$$

$$G_z = \dot{r}I_z + pq(I_y - I_x) + (qr - \dot{p})I_{xz} \quad (2.48)$$

This completes the development of the RHS of the six equations listed on page 2.6. Remember the RHS is the aircraft response or the motion of the aircraft that would result from the application of a force or a moment. The LHS of the equation represents these applied forces or moments.

## 2.6 LEFT-HAND SIDE OF EQUATION

### 2.6.1 Terminology

Before launching into the development of the LHS, it will help to clarify some of the terms used to describe the motion of the aircraft.

Steady Flight - Flight in which the existing motion remains steady with time, i.e., no transient conditions exist.

Symmetric Flight - (Longitudinal Motion) - Flight in which the vehicle plane of symmetry remains fixed in space.

$$v = 0 \quad p = r = 0$$

$$(\beta = 0) \quad (\phi \text{ and } \dot{\psi} = 0)$$

Asymmetric Flight - (Lateral Motion) - Flight in which the vehicle plane of symmetry does not remain fixed in space.

$$v \neq 0 \quad p \text{ and/or } r \neq 0$$

$$(\beta \neq 0) \quad (\phi \text{ and/or } \dot{\psi} \neq 0)$$

### 2.6.2 Some Special-Case Vehicle Motions

#### Unaccelerated Flight

(Also called straight flight or equilibrium flight.)

$$F_x = 0 \quad F_y = 0 \quad F_z = 0$$

Hence, the cg travels a straight path at constant speed. Note that equilibrium does not mean steady state. For example,

$$F_x = m(\dot{u} + qw - rv) = 0$$

could be maintained zero by fluctuation of the three terms on the right in an unsteady manner. In practice, however, it is difficult to predict that non-steady motion will remain unaccelerated and hence the straight motions most often discussed are also steady state.

<u>Steady Straight Flight</u>	<u>Steady Rolls or Spins</u>
$F_x = 0$ $F_y = 0$ $F_z = 0$ $G_x = 0$ $G_y = 0$ $G_z = 0$ <u>On the average</u>	$F_x = 0$ $F_y = 0$ $F_z = 0$ $G_x = 0$ $G_y = 0$ $G_z = 0$ <u>On the average</u>
$p = q = r = 0$ <u>Excluded by custom</u>	By custom this is not called straight flight even though the cg may be traveling a straight path
Trim Points, Stabilized Points	Steady Developed Spins

### Accelerated Flight (Non-Equilibrium Flight)

One or more of the linear equations is not zero, hence the cg is not traveling a straight path. Again the steady cases are of most interest.

#### Steady Turns

An unbalanced horizontal force results in the cg being constantly deflected inward toward the center of a curved path. This results in a constantly changing yaw angle. By the Euler angle transform,

$$p = -\dot{\psi}\theta \text{ (assumes small } \theta \text{)}$$

$$q = \dot{\psi} \sin \phi \cos \theta = \dot{\psi} \sin \phi$$

$$r = \dot{\psi} \cos \phi \cos \theta = \dot{\psi} \cos \phi$$

and hence

$$F_y = m (\dot{\psi} \cos \phi)u$$

$$F_z = -m (\dot{\psi} \sin \phi)u \quad \text{(assumes } \dot{\psi}\theta \text{ is very very small)}$$

Includes moderate climbs and descents.

#### Symmetrical Pull Up

Here an unbalanced  $z$  force is constantly deflecting the cg upward.

$$q = \dot{\theta}$$

$$F_x \approx mgw$$

and

$$F_z \approx -mqw$$

This is a quasi-steady motion since  $\dot{u}$  and  $\dot{w}$  cannot long remain zero.

### 2.6.3 Preparation for Expansion of the Left-Hand Side

The Equations of Motion relate the vehicle motion to the applied forces and moments.

<div style="text-align: center; margin-bottom: 5px;">— LHS —</div> <div style="text-align: center;">Applied Forces and Moments</div>	=	<div style="text-align: center; margin-bottom: 5px;">— RHS —</div> <div style="text-align: center;">Observed Vehicle Motion</div>
$F_x = m\dot{u} + \text{---}$ $G_x = \dot{p}I_x + \text{---}$ <p>etc.</p>		

The RHS of each of these six equations has been completely expanded in terms of easily measured quantities. The LHS must also be expanded in terms of convenient variables, to include Stability Parameters and Derivatives. Before this can be accomplished, however, the following topics must be discussed and understood.

## 2.6.4 Initial Breakdown of the Left-Hand Side

In general, the applied forces and moments can be broken up according to the sources shown below.

		Source					
		Aero-dynamic	Direct Thrust	Gravity	Gyro-scopic	Other	
LONGITUDINAL	$F_x$	$X$	$X_T$	$X_g$	0	$X_{oth}$	$= m\dot{u} + - - - (2.49)$
	$F_z$	$Z$	$Z_T$	$Z_g$	0	$Z_{oth}$	$= m\dot{w} + - - - (2.50)$
	$G_y$	$M$	$M_T$	0	$M_{gyro}$	$M_{oth}$	$= \dot{q}I_y + - - - (2.51)$
LATERAL-DIRECTIONAL	$F_y$	$Y$	$Y_T$	$Y_g$	0	$Y_{oth}$	$= m\dot{v} + - - - (2.52)$
	$G_x$	$L$	$L_T$	0	$L_{gyro}$	$L_{oth}$	$= \dot{p}I_x + - - - (2.53)$
	$G_z$	$N$	$N_T$	0	$N_{gyro}$	$N_{oth}$	$= \dot{r}I_z + - - - (2.54)$

1. Gravity Forces - These vary with orientation of the weight vector.  

$$X_g = -mg \sin \theta \quad Y_g = mg \cos \theta \sin \phi \quad Z_g = mg \cos \theta \cos \phi$$
2. Gyroscopic Moments - These occur as a result of large rotating masses such as engines and props.
3. Direct Thrust Forces and Moments - These terms include the effect of the thrust vector itself - they usually do not include the indirect or induced effects of jet flow or running propellers.
4. Aerodynamic Forces and Moments - These will be further expanded into Stability Parameters and Derivatives.
5. Other Sources - These include spin chutes, reaction controls, etc.

## 2.6.5 Aerodynamic Forces and Moments

By far the most important forces and moments on the LHS of the equation are the aerodynamic terms. Unfortunately they are also the most complex. As a result, certain simplifying assumptions are made and several of the smaller terms are arbitrarily excluded to simplify the analysis. Remember we are not trying to design an airplane around some critical criteria. We are only trying to derive a set of equations that will help us analyze the important factors affecting aircraft stability and control.

### Choice of Axis System

Consider the aerodynamic forces on an airplane

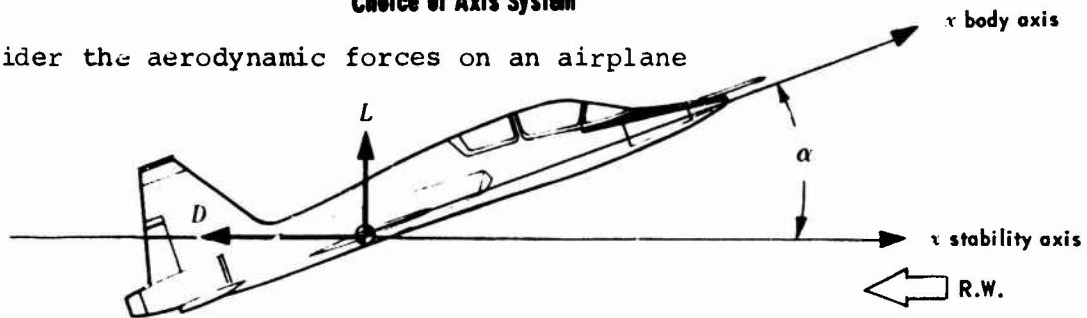


Figure 2.19

Summing forces along the  $x$  body axis

$$F_x = L \sin \alpha - D \cos \alpha \quad (2.55)$$

Notice that if the forces were summed along the  $x$  stability axis, it would be

$$F_x = -D \quad (2.56)$$

Obviously, it would simplify things if the stability axes were used for development of the aerodynamic forces. A small angle assumption will enable us to do this. Let's assume that  $\alpha$  is always small enough so that

$$\cos \alpha \approx 1$$

$$\sin \alpha \approx 0$$

using this assumption, equation 2.55 reduces to equation 2.56. Whether it be thought of as a small angle assumption or as an arbitrary choice of the stability axis system, the result is the same. The complexity of the equation is reduced. This of course would not be done for preliminary design analyses, however, for the purpose of deriving a set of equations to be used as an analytical tool in determining handling qualities, the assumption is perfectly valid, and in fact, is surprisingly accurate for relatively large values of  $\alpha$ .

Therefore, the aerodynamic terms will be developed using the stability axis system so that the equations assume the form,

$$\text{"DRAG"} \quad -D + X_T + X_g + X_{oth} = m\dot{u} + - - - - \quad (2.57)$$

$$\text{"LIFT"} \quad -L + Z_T + Z_g + Z_{oth} = m\dot{w} + - - - - \quad (2.58)$$

$$\text{"PITCH"} \quad M + M_T + M_{gyro} + M_{oth} = \dot{q}I_y + - - - - \quad (2.59)$$

$$\text{"SIDE"} \quad Y + Y_T + Y_g + Y_{oth} = m\dot{v} + - - - - \quad (2.60)$$

$$\text{"ROLL"} \quad L + L_T + L_{gyro} + L_{oth} = \dot{p}I_x + - - - - \quad (2.61)$$

$$\text{"YAW"} \quad N + N_T + N_{gyro} + N_{oth} = \dot{r}I_z + - - - - \quad (2.62)$$

## Expansion of Aerodynamic Terms

A stability and control analysis is concerned with the question of how a vehicle responds to certain perturbations or inputs. For instance, up elevator should cause the nose to come up; or if the aircraft hits some turbulence that causes a small amount of sideslip, the airplane should realign itself with the relative wind. Intuitively, it can be seen that the aerodynamic terms are going to have the most effect on the resulting motion of the aircraft. Unfortunately, the equations that result from summing forces and moments are non-linear. As a result, exact solutions to these equations are impossible. Therefore, a technique to linearize the equations must be used so that solutions can be obtained. In order to do this the small perturbation theory is introduced.

### Small Perturbation Theory

The small perturbation theory is based on a simple and very popular technique used for linearizing a set of differential equations. In a nutshell, it is simply the process of expanding the equations using a Taylor series expansion and excluding the higher order terms. To fully understand the derivation some assumptions and definitions must first be established.

#### The Small Disturbance Assumption

The aerodynamic forces and moments are primarily a function of the following variables:

1. Temperature and Altitude

Accounted for by  $\rho$ ,  $M$ ,  $R_e$ .

2. Angular Velocities

Accounted for by  $p$ ,  $q$ ,  $r$ .

3. Control Deflections

Accounted for by  $\delta_e$ ,  $\delta_a$ ,  $\delta_r$ .

4. Position and Magnitude of the Relative Wind

Accounted for by the components  $u$ ,  $v$ ,  $w$  of true velocity, or alternately, by:

$$u, \quad \alpha \approx \frac{w}{V_T}, \quad \beta \approx \frac{v}{V_T}$$

In general, the time derivatives of these variables could also be significant. In other words:

		VARIABLE	FIRST DERIVATIVE	SECOND DERIVATIVE
$\left. \begin{matrix} D \\ L \\ M \\ Y \\ \mathcal{L} \\ \mathcal{N} \end{matrix} \right\}$	Are a Function of	$u \ \alpha \ \beta$	$\dot{u} \ \dot{\alpha} \ \dot{\beta}$	$\ddot{u} \ - \ - \ - \ -$
		$p \ q \ r$	$\dot{p} \ - \ -$	$\ddot{p} \ - \ - \ - \ -$
		$\delta_e \ \delta_a \ \delta_r$	$\dot{\delta}_e \ - \ -$	$- \ - \ -$
		$\rho \ M \ R_e \rightarrow$	assumed constant	

This rather formidable list can be reduced to workable proportions by making the assumption that the vehicle motion will consist only of small deviations from some initial reference condition. Fortunately, this small disturbance assumption applies to many cases of practical interest, and as a bonus, stability parameters and derivatives derived under this assumption continue to give good results for motions somewhat larger.

The variables are considered to consist of some initial value plus an incremental change, called the "perturbated value." The notation for these perturbated values is sometimes lower case and sometimes lower case with a bar.

$$\begin{aligned} P &= P_0 + p & p &= p_0 + \bar{p} \\ U &= U_0 + u & u &= u_0 + \bar{u} \end{aligned}$$

It has been found from experience that when operating under the small disturbance assumption the vehicle motion can be thought of as two independent motions each of which is a function only of the variables shown below.

#### 1. Longitudinal Motion

$$(D, L, M) = f(U, \alpha, \dot{\alpha}, Q, \delta_e) \quad (2.63)$$

#### 2. Lateral-Directional Motion

$$(Y, \mathcal{L}, \mathcal{N}) = g(\beta, \dot{\beta}, P, R, \delta_a, \delta_r) \quad (2.64)$$

#### Initial Conditions

It will be assumed that the motion consists of small perturbations about some initial condition of steady straight symmetrical flight. From this and the definition of stability axes, the following can be stated:

$$V_T = U_O = \text{constant}$$

$$V_O = 0$$

$$\beta_O = 0$$

$$W_O = 0$$

$$\alpha_O = \text{constant}$$

$$P_O = Q_O = R_O = 0$$

$$(\rho, M, R_e, \text{aircraft configuration.}) = \text{constant}$$

### 2.6.6 Expansion by Taylor Series

As previously noted, the longitudinal motion can be assumed to be a function of five variables,  $U$ ,  $\alpha$ ,  $\dot{\alpha}$ ,  $Q$ , and  $\delta_e$ . The aerodynamic forces and moments can therefore be expressed by a Taylor's expansion.

For example:

$$L = \begin{bmatrix} L_O + \frac{\partial L}{\partial U} \Delta U + \frac{1}{2} \frac{\partial^2 L}{\partial U^2} \Delta U^2 + - - - \\ + \frac{\partial L}{\partial \alpha} \Delta \alpha + \frac{1}{2} \frac{\partial^2 L}{\partial \alpha^2} \Delta \alpha^2 + - - - \\ + \frac{\partial L}{\partial \dot{\alpha}} \Delta \dot{\alpha} + - - - - \\ + \frac{\partial L}{\partial Q} \Delta Q + - - - - \\ + \frac{\partial L}{\partial \delta_e} \Delta \delta_e + - - - - \end{bmatrix} \quad (2.65)$$

But we have decided to express the variables as the sum of an initial value plus a small perturbed value

$$U = U_O + u \quad \text{where} \quad u = U - U_O = \Delta U \quad (2.66)$$

Therefore

$$\frac{\partial L}{\partial U} = \frac{\partial L}{\partial U_O} \cdot \frac{\partial U_O}{\partial U} \overset{\text{Zero}}{\cancel{= 0}} + \frac{\partial L}{\partial u} \frac{\partial u}{\partial U} \overset{1.0}{\cancel{= 1}} = \frac{\partial L}{\partial u} \quad (2.67)$$

and the first term of the expansion becomes

$$\frac{\partial L}{\partial U} \Delta U = \frac{\partial L}{\partial u} u \quad (2.68)$$

Similarly:

$$\frac{\partial L}{\partial Q} \Delta Q = \frac{\partial L}{\partial q} q \quad (2.69)$$

We also elect to let  $\alpha = \Delta\alpha$ ,  $\dot{\alpha} = \Delta\dot{\alpha}$  and  $\delta_e = \Delta\delta_e$ .

Dropping higher order terms involving  $u^2$ ,  $q^2$ , etc., equation 2.65 now becomes

$$L = L_0 + \frac{\partial L}{\partial u} u + \frac{\partial L}{\partial \alpha} \alpha + \frac{\partial L}{\partial \dot{\alpha}} \dot{\alpha} + \frac{\partial L}{\partial q} q + \frac{\partial L}{\partial \delta_e} \delta_e \quad (2.70)$$

The lateral-directional motion is a function of  $\beta$ ,  $\dot{\beta}$ ,  $p$ ,  $r$ ,  $\delta_a$ ,  $\delta_r$ , and can be handled in a similar manner. For example, the aerodynamic terms for rolling moment become:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_0 &+ \frac{\partial \mathcal{L}}{\partial \beta} \beta + \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \dot{\beta} + \frac{\partial \mathcal{L}}{\partial p} p + \frac{\partial \mathcal{L}}{\partial r} r + \frac{\partial \mathcal{L}}{\partial \delta_a} \delta_a \\ &+ \frac{\partial \mathcal{L}}{\partial \delta_r} \delta_r \end{aligned} \quad (2.71)$$

This development can be applied to all of the aerodynamic forces and moments. The equations are linear and account for all variables that have a significant effect on the aerodynamic forces and moments on an aircraft.

The equations resulting from this development can now be substituted into the LHS of the equations of motion.

### 2.6.7 Effects of Weight

The weight force acts through the cg of an airplane and as a result has no effect on the aircraft moments. It does affect the force equations as shown below.

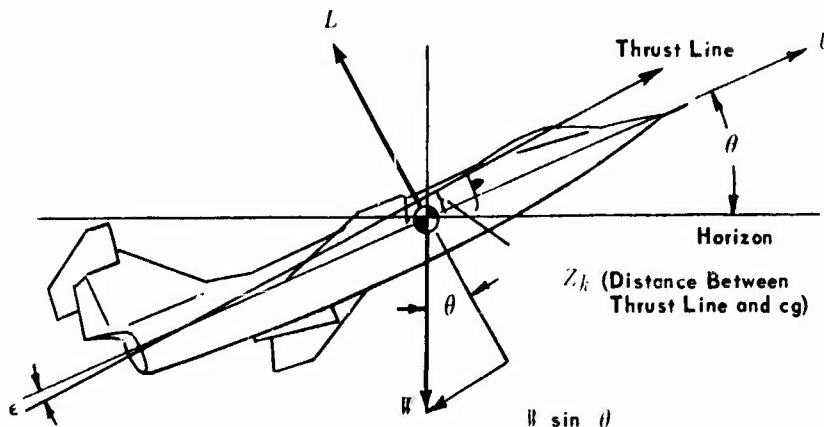


Figure 2.20

The same "small perturbation" technique can be used to analyze the effects of weight. For longitudinal motion, the only variable to consider is  $\theta$ . For example, consider the effect of weight on the  $x$  axis.

$$X_g = -W \sin \theta \quad (2.72)$$

Since weight is considered constant,  $\theta$  is the only pertinent variable. Therefore, the expansion of the gravity term ( $X_g$ ) can be expressed as

$$X_g = X_{g_0} + \frac{\partial X_g}{\partial \theta} \theta \quad (X_{g_0} = \text{initial condition of } X_g) \quad (2.73)$$

For simplification and clarity, the term  $X_g$  will hereafter be referred to as drag due to weight, ( $D_{wt}$ ). This in essence incorporates the same small angle assumption that was made in development of the aerodynamic terms, however, as before, the effect is negligible. Therefore, equation 2.73 becomes

$$D_{wt} = D_{0wt} + \frac{\partial D}{\partial \theta} \theta$$

Likewise the  $Z$  force can be expressed as negative lift due to weight ( $L_{wt}$ ), and the expanded term becomes

$$L_{wt} = L_{0wt} + \frac{\partial L}{\partial \theta} \theta$$

The effect of weight on side force is dependent solely on bank angle ( $\phi$ ). Therefore,

$$Y_{wt} = Y_{0wt} + \frac{\partial Y}{\partial \phi} \phi$$

These then are the component equations relating the effects of gravity to the equations of motion and can be substituted into the LHS of the equations.

#### 2.6.8 Effects of Thrust

The thrust vector can be considered in the same way. Since thrust does not always pass through the cg its effect on the moment equation must be considered (figure 2.20). The  $X$  component would be

$$X_T = T \cos \epsilon$$

The  $Z$  component is

$$Z_T = -T \sin \epsilon$$

The pitching moment component is,

$$M_T = T Z_k$$

where  $Z_k$  is the perpendicular distance from the thrust line to the cg. For small disturbances, changes in thrust depend upon the change in forward speed and engine rpm. Therefore, by a small perturbation analysis

$$T = T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm} \quad (2.74)$$

Thrust effects will be considered in the longitudinal equations only since the thrust vector is normally in the vertical plane of symmetry and does not affect the lateral-directional motion. When considering engine-out characteristics in multi-engine aircraft; however, the asymmetric thrust effects must be considered. Once again, for clarity,  $X_T$  and  $Z_T$  will be referred to as drag due to thrust and lift due to thrust.

Thus:

$$D_{THRUST} = (T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}) (\cos \epsilon) \quad (2.75)$$

$$L_{THRUST} = -(T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}) (\sin \epsilon) \quad (2.76)$$

$$M_{THRUST} = (T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm}) (Z_k) \quad (2.77)$$

## 2.6.9 Gyroscope Effects

For most analyses, gyroscopic effects are insignificant. They begin to become important as angular rates increase, (i.e.,  $p$ ,  $q$ , and  $r$  become large). For static and dynamic stability analyses, angular rates are not considered large. In the area of spins and maximum roll rate maneuvers, they are large and definitely affect the motion of the airplane. Therefore, for spin and roll coupling analyses, gyroscopic effects will be considered. However, in the basic development of the equations of motion, they will not be included.

## 2.7 REDUCTION OF EQUATIONS TO A USABLE FORM

### 2.7.1 Normalization of Equations

Now that the linearized expressions have been derived, we begin the process of putting them in a more usable form. One of the first steps in this process is to "normalize" the equations. Initially, the reason for doing this will not be apparent. It is a necessary step in the simplification of the equations, however, and the rationale will become apparent later.

In order to do this each equation is multiplied through by a "normalization factor". This factor is different for each equation and is picked primarily to simplify the first term on the RHS of the equation. The following table presents the normalizing factor for each equation.

Equation	Normalizing Factor	First Term is Now Pure Accel or $\dot{\alpha}$ $\dot{\beta}$	Units
"DRAG"	$\frac{1}{m}$	$-\frac{D}{m} + \frac{X_T}{m} + \dots = \dot{u}$	$[\frac{ft}{sec^2}]$ (2.78)
"LIFT"	$\frac{1}{mU_o}$	$-\frac{L}{mU_o} + \frac{Z_T}{mU_o} + \dots = \frac{\dot{w}}{U_o} + \dots$	$[\frac{rad}{sec}]$ (2.79)
"PITCH"	$\frac{1}{I_y}$	$\frac{M}{I_y} + \frac{M_T}{I_y} + \dots = \dot{q}$	$[\frac{rad}{sec^2}]$ (2.80)
"SIDE"	$\frac{1}{mU_o}$	$\frac{Y}{mU_o} + \frac{Y_T}{mU_o} + \dots = \frac{\dot{v}}{U_o} + \dots$	$[\frac{rad}{sec}]$ (2.81)
"ROLL"	$\frac{1}{I_x}$	$\frac{L}{I_x} + \frac{L_T}{I_x} + \dots = \dot{p} + \dots$	$[\frac{rad}{sec^2}]$ (2.82)
"YAW"	$\frac{1}{I_z}$	$\frac{N}{I_z} + \frac{N_T}{I_z} + \dots = \dot{r} + \dots$	$[\frac{rad}{sec^2}]$ (2.83)

## 2.7.2 Stability Parameters

Stability parameters are simply the partial coefficients ( $\frac{\partial L}{\partial u}$ , etc.) multiplied by their respective normalizing factors. To demonstrate this, consider the aerodynamic terms of the lift equation. By multiplying equation 2.70 through by the factor  $\frac{1}{mU_o}$ , we get

$$\frac{L}{mU_o} = \frac{L_o}{mU_o} + \underbrace{\frac{1}{mU_o} \frac{\partial L}{\partial u}}_{L_u} u + \underbrace{\frac{1}{mU_o} \frac{\partial L}{\partial \alpha}}_{L_\alpha} \alpha + \underbrace{\frac{1}{mU_o} \frac{\partial L}{\partial \dot{\alpha}}}_{L_{\dot{\alpha}}} \dot{\alpha} + \underbrace{\frac{1}{mU_o} \frac{\partial L}{\partial q}}_{L_q} q + \underbrace{\frac{1}{mU_o} \frac{\partial L}{\partial \delta_e}}_{L_{\delta_e}} \delta_e \quad [\frac{rad}{sec}] \quad (2.84)$$

The indicated quantities are defined as stability parameters and the equation becomes

$$\frac{L}{mU_o} = \frac{L_o}{mU_o} + L_u u + L_\alpha \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_q q + L_{\delta_e} \delta_e \quad [\frac{rad}{sec}] \quad (2.85)$$

Stability parameters have various dimensions depending on whether they are multiplied by a linear velocity, an angle, or an angular rate

$$L_u \left[ \frac{1}{ft} \right] u \left[ \frac{ft}{sec} \right] = \left[ \frac{rad}{sec} \right], \quad L_\alpha \left[ \frac{1}{sec} \right] \alpha \left[ rad \right] = \left[ \frac{rad}{sec} \right],$$

$$L_{\dot{\alpha}} \left[ none \right] \dot{\alpha} \left[ \frac{rad}{sec} \right] = \left[ \frac{rad}{sec} \right]$$

The lateral-directional motion can be handled in a similar manner. For example, the normalized aerodynamic rolling moment becomes:

$$\frac{L}{I_x} = \frac{L_0}{I_x} + L_{\beta} \beta + L_{\dot{\beta}} \dot{\beta} + L_p p + L_r r + L_{\delta_a} \delta_a + L_{\delta_r} \delta_r \left[ \frac{1}{sec^2} \right] \quad (2.86)$$

where

$$L_{\beta} = \frac{1}{I_x} \frac{\partial L}{\partial \beta} \left[ \frac{1}{sec^2} \right], \text{ etc.}$$

These stability parameters are sometimes called "dimensional derivatives" but we will reserve the word "derivative" to indicate the non-dimensional form which can be obtained by rearrangement. This will be developed later in this chapter.

### 2.7.3 Simplification of the Equations

By combining all of the terms derived so far, the resulting equations are somewhat lengthy. In order to economize on effort, several simplifications can be made. For one, all "small effect" terms can be disregarded. Normally these terms are an order of magnitude less than the more predominant terms. These and other simplifications will help derive a concise and workable set of equations.

### 2.7.4 Longitudinal Equations

#### Drag Equation

The complete normalized drag equation is

$$\begin{aligned} & \text{Aero Terms} & \text{Gravity Terms} \\ & - \left[ \frac{D_0}{m} + D_{\dot{\alpha}} \dot{\alpha} + D_{\dot{\beta}} \dot{\beta} + D_u u + D_q q + D_{\delta_e} \delta_e \right] & - \left[ \frac{D_{0wt}}{m} + D_{\mu} \mu \right] \\ & \text{Thrust Terms} \\ & + \frac{1}{m} \left[ T_0 + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm} \right] (\cos \epsilon) = \dot{u} + q - rv \quad (2.87) \end{aligned}$$

## Simplifying assumptions

1.  $\frac{T_o}{m} \cos \epsilon - \frac{D_o}{m} - \frac{D_{o_{wt}}}{m} = 0$  (Steady State)
  2.  $\left( \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta} \delta_{rpm} \right) \cos \epsilon = 0$  (Constant rpm,  $\frac{\partial T}{\partial u}$  is small)
  3.  $rv = 0$  (No lat-dir motion)
- The small perturbation assumption allows us to analyze the longitudinal motion independent of lateral-directional motion.
4.  $qw \approx 0$  (Order of magnitude)
  5.  $D_{\alpha}$ ,  $D_q$ , and  $D_{\delta_e}$  are all small.

The resulting equation is

$$-[D_{\alpha} \alpha + D_u u + D_{\theta} \theta] = \dot{u} \quad (2.88)$$

Rearranging

$$\dot{u} + D_{\alpha} \alpha + D_u u + D_{\theta} \theta = 0 \quad (2.89)$$

## Lift Equation

The complete lift equation is

$$\begin{aligned}
 & \underbrace{- \left[ \frac{L_o}{mU_o} + L_{\alpha} \alpha + L_{\dot{\alpha}} \dot{\alpha} + L_u u + L_q q + L_{\delta_e} \delta_e \right]}_{\text{Aero Terms}} + \underbrace{\left[ \frac{L_{o_{wt}}}{mU_o} + L_{\theta} \theta \right]}_{\text{Gravity Terms}} \\
 & \underbrace{- \frac{1}{mU_o} \left[ T_o + \frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta} \delta_{rpm} \right]}_{\text{Thrust Terms}} (\sin \epsilon) = \frac{\dot{w} + pv - qu}{U_o}
 \end{aligned}$$

### Simplifying assumptions

1.  $-\frac{L_o}{mU_o} + \frac{L_{owt}}{mU_o} - \frac{T_o}{mU_o} \sin \epsilon = 0$  (Steady State)
2.  $\frac{\partial T}{\partial u} u + \frac{\partial T}{\partial \delta_{rpm}} \delta_{rpm} = 0$  (Constant rpm,  $\frac{\partial T}{\partial u}$  is small)
3.  $L_\theta \theta = 0$  (Order of magnitude)
4.  $\frac{\dot{w}}{U_o} = \dot{\alpha}$
5.  $pv = 0$  (No lat-dir motion)
6.  $\frac{qu}{U_o} = q$  ( $u = U_o$ )

Thus

$$-L_\alpha \alpha - L_{\dot{\alpha}} \dot{\alpha} - L_u u - L_q q - L_{\delta_e} \delta_e = \dot{\alpha} - q \quad (2.90)$$

Rearranging

$$-L_\alpha \alpha - (1 + L_{\dot{\alpha}}) \dot{\alpha} - L_u u + (1 - L_q) q = L_{\delta_e} \delta_e \quad (2.91)$$

### Pitch Moment Equation

$$\begin{aligned} \frac{M_o}{I_y} + M_{\alpha} \alpha + M_{\dot{\alpha}} \dot{\alpha} + M_u u + M_{\delta_e} \delta_e + M_q q + \text{Thrust Terms} \\ = \dot{q} - pr \frac{(I_z - I_x)}{I_y} + \frac{(p^2 - r^2)}{I_y} I_{xy} \end{aligned} \quad (2.92)$$

This can be simplified as before. Thus

$$\dot{q} - M_{\alpha} \alpha - M_{\dot{\alpha}} \dot{\alpha} - M_u u - M_q q = M_{\delta_e} \delta_e \quad (2.93)$$

Now there are three longitudinal equations that are easy to work with. Notice that there are four variables  $\alpha$ ,  $\dot{\alpha}$ ,  $u$ , and  $q$  but only three equations. To solve this problem,  $\dot{\alpha}$  can be substituted for  $q$ .

$$q = \dot{\alpha} \quad \text{and} \quad \dot{q} = \ddot{\alpha}$$

Therefore the longitudinal equations become:

$$\begin{array}{c|c|c|c|c} & (\theta) & (u) & (\alpha) & \\ \hline \text{DRAG} & D_{\theta} \dot{\theta} & + \dot{u} + D_u u & + D_{\alpha} \dot{\alpha} & = 0 \end{array} \quad (2.94)$$

$$\begin{array}{c|c|c|c|c} & (1-L_q) \dot{\theta} & - L_u u & - (1 + L_{\alpha}) \dot{\alpha} - L_{\alpha} \alpha & = L_{\delta_e} \delta_e \end{array} \quad (2.95)$$

$$\begin{array}{c|c|c|c|c} \text{PITCH} & \ddot{\theta} - \mathcal{M}_q \dot{\theta} & - \mathcal{M}_u u & - \mathcal{M}_{\alpha} \dot{\alpha} - \mathcal{M}_{\alpha} \alpha & = \mathcal{M}_{\delta_e} \delta_e \end{array} \quad (2.96)$$

There are now three independent equations with three variables. The terms on the RHS are the inputs or "forcing functions". Therefore, for any input  $\delta_e$ , the equations can be solved to get  $\theta$ ,  $u$  and  $\alpha$  at any time.

#### Lateral-Directional Equations

The complete lateral-directional equations are as follows:

Side Force

$$\begin{aligned} \frac{Y_o}{mU_o} + Y_{\beta} \beta + Y_{\dot{\beta}} \dot{\beta} + Y_p p + Y_r r + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r \\ + \frac{Y_{o_{wt}}}{mU_o} + Y_{\phi} \phi = \frac{\dot{v} + ru - pw}{U_o} \end{aligned} \quad (2.97)$$

Rolling Moment

$$\begin{aligned} \frac{L_o}{I_x} + L_{\beta} \beta + L_{\dot{\beta}} \dot{\beta} + L_p p + L_r r + L_{\delta_a} \delta_a + L_{\delta_r} \delta_r \\ = \dot{p} + qr \left( \frac{I_z - I_y}{I_x} \right) - (\dot{r} - pq) \frac{I_{xz}}{I_x} \end{aligned} \quad (2.98)$$

Yawing Moment

$$\begin{aligned} \frac{N_o}{I_z} + N_{\beta} \beta + N_{\dot{\beta}} \dot{\beta} + N_p p + N_r r + N_{\delta_a} \delta_a + N_{\delta_r} \delta_r \\ = \dot{r} + pq \left( \frac{I_y - I_x}{I_z} \right) + (qr - \dot{p}) \frac{I_{xz}}{I_z} \end{aligned} \quad (2.99)$$

In order to simplify the equations, the following assumptions are made:

1. A wings level steady state condition exists initially. Therefore,  $L_0$ ,  $N_0$ ,  $Y_0$ , and  $Y_{0wt}$  are zero.
2.  $p = \dot{\phi}$ ,  $\dot{p} = \ddot{\phi}$  ( $\theta \approx 0$ , see Euler angle transformations, pg 2.7)
3. The terms  $Y_{\dot{\beta}}\dot{\beta}$ ,  $N_{\dot{\beta}}\dot{\beta}$ ,  $L_{\dot{\beta}}\dot{\beta}$  and  $Y_{\delta_a}\delta_a$  are all small.
4.  $\frac{\dot{v}}{U_0} = \dot{\beta}$  ( $\alpha$  is small)
5.  $\frac{ru}{U_0} = r$  ( $u = U_0$ )
6.  $w = q = 0$  (no longitudinal motion)

Using these assumptions the lateral-directional equations reduce to:

$$\begin{array}{c|c|c|c} (\beta) & (\dot{\phi}) & (r) & \\ \hline \text{SIDE} & \dot{\beta} - Y_{\dot{\beta}}\dot{\beta} & - Y_p\dot{\phi} - Y_{\dot{\phi}}\dot{\phi} & + (1 - Y_r)r \\ \text{FORCE} & & & = Y_{\delta_r}\delta_r \end{array} \quad (2.100)$$

$$\begin{array}{c|c|c|c} \text{ROLLING} & - L_{\dot{\beta}}\dot{\beta} & + \ddot{\phi} - L_p\dot{\phi} & - \frac{I_{xz}}{I_x}\dot{r} - L_r r \\ \text{MOMENT} & & & = L_{\delta_a}\delta_a + L_{\delta_r}\delta_r \end{array} \quad (2.101)$$

$$\begin{array}{c|c|c|c} \text{YAWING} & - N_{\dot{\beta}}\dot{\beta} & - \frac{I_{xz}}{I_z}\ddot{\phi} - N_p\dot{\phi} & + \dot{r} - N_r r \\ \text{MOMENT} & & & = N_{\delta_a}\delta_a + N_{\delta_r}\delta_r \end{array} \quad (2.102)$$

Once again there are three unknowns and three equations. These equations may be used to analyze the lateral-directional motion of the aircraft.

### 2.7.5 Stability Derivatives

The parametric equations give all the information necessary to describe the motion of any particular airplane. There is only one problem. When using a wind tunnel model for verification, a scaling factor must be used to find the values for the aircraft. In order to eliminate this requirement a set of non-dimensional equations must be derived. This can be illustrated best by an example:

# EXAMPLE

Given the parametric equation for pitching moment,

$$\ddot{\theta} - \mathcal{M}_q \dot{\theta} - \mathcal{M}_u u - \mathcal{M}_{\dot{a}} \dot{a} - \mathcal{M}_{\dot{a}} \dot{a} = \mathcal{M}_{\delta_e} \delta_e$$

Derive an equation in which all terms are NON-DIMENSIONAL.

The steps in this process are

1. Take each stability parameter and substitute its coefficient relation, i.e.,

$$\mathcal{M}_q = \frac{1}{I_y} \frac{\partial \mathcal{M}}{\partial q} = \frac{1}{I_y} \frac{\partial (C_m \frac{1}{2} \rho U^2 S c)}{\partial q} \quad (2.103)$$

$C_m$  is the only variable that is dependent on  $q$ , therefore,

$$\mathcal{M}_q = \frac{\rho U_o^2 S c}{2 I_y} \left( \frac{\partial C_m}{\partial q} \right) \quad (2.104)$$

2. Non-dimensionalize the partial term, i.e.,

$$\frac{\partial C_m}{\partial q} \text{ has dimensions} = \frac{\text{dimensionless}}{\text{rad/sec}} = \text{sec}$$

To non-dimensionalize the partial terms, there exist certain compensating factors that will be shown later. In this case the compensating factor is

$$\frac{c}{2U_o} \frac{[ft]}{[ft/sec]} = \text{sec}$$

Multiply and divide equation 2.104 by the compensating factor and get

$$\mathcal{M}_q = \frac{\rho U_o^2 S c}{2 I_y} \cdot \frac{\frac{c}{2U_o}}{\frac{c}{2U_o}} \cdot \left( \frac{\partial C_m}{\partial q} \right) \rightarrow \text{This term is now dimensionless}$$

check

$$\frac{\frac{\partial C_m}{\partial (\frac{cq}{2U_o})}}{\frac{ft/sec}{ft/sec}} = \frac{\text{dimensionless}}{\text{dimensionless}} = \text{dimensionless}$$

This is called a stability derivative and is written

$$C_{m_q} = \frac{\partial C_m}{\partial \left(\frac{cq}{2U_o}\right)} - \text{by definition}$$

3. When the entire term as originally derived is considered, i.e.,

$$M_{q^q} = \frac{\rho U_o^2 S c}{2I_y} \cdot \frac{c}{2U_o} \cdot C_{m_q} \cdot q$$

It can be rearranged so that

$$M_{q^q} = \frac{\rho U_o^2 S c}{2I_y} \cdot C_{m_q} \cdot \frac{cq}{2U_o}$$

Define

$$\hat{q} = \frac{cq}{2U_o} \frac{[ft/sec]}{[ft/sec]} \text{ dimensionless}$$

∴ The term becomes

$$M_{q^q} = \frac{\rho U_o^2 S c}{2I_y} C_{m_q} \hat{q}$$

$\xrightarrow{\quad}$  Dimensionless variable  
 $\xrightarrow{\quad}$  Dimensionless stability derivative  
 $\xrightarrow{\quad}$  Constants

But  $q$  is expressed as  $\dot{\theta}$  in the equation. To convert this substitute  $\frac{d\theta}{dt}$  for  $q$ .

$$\hat{q} = \frac{cq}{2U_o} = \frac{c}{2U_o} \frac{d(\theta)}{dt}$$

Then,

$$\text{Define } D = \frac{c}{2U_o} \frac{d(\quad)}{dt}$$

$D$  can be considered to be a dimensionless derivative and acts like an operator.

therefore

$$\hat{q} = D \theta = \frac{c}{2U_0} \frac{d\theta}{dt} \text{ (i.e., dimensionless derivative of } \theta \text{)}$$

4. Do the same for each term in the parametric equation.

$$m_u = \frac{1}{I_y} \frac{\partial m}{\partial u} = \frac{1}{I_y} \frac{\partial (C_m \frac{1}{2} \rho U^2 Sc)}{\partial u}$$

Since both  $C_m$  and  $U$  are functions of  $u$ , then

$$m_u = \frac{\rho U_0^2 Sc}{2I_y} \left( \frac{\partial C_m}{\partial u} + \frac{2C_{m0}}{U_0} \right)$$

but  $C_{m0} = 0$  since initial conditions are steady state. The compensating factor for this case is  $\frac{1}{U_0}$

$$\therefore m_{u^u} = \frac{\rho U_0^2 Sc}{2I_y} \cdot \left( \frac{\partial C_m}{\partial \frac{u}{U_0}} \right) \cdot \left( \frac{u}{U_0} \right)$$

$\downarrow$   $\downarrow$   
 $C_{m_u}$   $\hat{u}$

5. Once all of the terms have been derived, they are substituted into the original equation, and multiplied through by

$$\frac{2I_y}{\rho U_0^2 Sc}$$

which gives

$$\frac{2I_y}{\rho U_0^2 Sc} \ddot{\theta} - C_{m_q} D\theta - C_{m_u} \hat{u} - C_{m_i} D\dot{\theta} - C_{m_e} \theta = C_{m_e} \theta$$

The first term is non-dimensional, however, it can be changed to a more convenient form. Multiply and divide by

$$\begin{aligned}
 & \frac{c^2}{4U_o^2} \left( \frac{2I_y}{\rho U_o^2 S c} \frac{4U_o^2}{c^2} \right) \cdot \left( \frac{c^2}{4U_o^2} \ddot{\theta} \right) \\
 & \rightarrow \text{Define } i_y = \frac{8I_y}{\rho S c^3} \\
 & \frac{c^2}{4U_o^2} \ddot{\theta} = \frac{c}{2U_o} \frac{d}{dt} \left( \frac{c}{2U_o} \frac{d(\theta)}{dt} \right) \\
 & = D^2 \theta
 \end{aligned}$$

Therefore, the term becomes,

$$i_y D^2 \theta$$

and the final equation is,

$$(i_y D^2 - C_{m_q}) \ddot{\theta} - C_{m_u} \dot{\theta} - (C_{m_o} D - C_{m_i}) \theta = C_{m_\delta} \delta_e$$

The compensating factors for all of the variables are listed below.

<u>Angular Rates</u>	<u>Compensating Factor</u>	<u>Non-Dimensional Angular Rates</u>
$p = \text{rad/sec}$	$\frac{b}{2U_o}$	$\hat{p} = \frac{pb}{2U_o} = D$
$q = \text{rad/sec}$	$\frac{c}{2U_o}$	$\hat{q} = \frac{qc}{2U_o} = D$
$r = \text{rad/sec}$	$\frac{b}{2U_o}$	$\hat{r} = \frac{rb}{2U_o} = D$

<u>Angular Rates</u>	<u>Compensating Factor</u>	<u>Non-Dimensional Angular Rates</u>
$\dot{\beta} = \text{rad/sec}$	$\frac{b}{2U_0}$	$\hat{\beta} = \frac{\dot{\beta} b}{2U_0} = D \beta$
$\dot{\alpha} = \text{rad/sec}$	$\frac{c}{2U_0}$	$\hat{\alpha} = \frac{\dot{\alpha} c}{2U_0} = D \alpha$
$u = \text{ft/sec}$	$\frac{1}{U_0}$	$\hat{u} = \frac{u}{U_0}$
$\alpha$ - no change		
$\beta$ - no change		

These derivations have been presented to give an understanding of where the equations come from and what they represent. It is not necessary to be able to derive each and every one of the equations. It is important, however, to understand several facts about the non-dimensional equations.

1. Since these equations are non-dimensional, they can be used to describe any aircraft that are geometrically similar.
2. Stability derivatives can be thought of as if they were stability parameters. Therefore,  $C_{m_{\dot{\alpha}}}$  refers to the same aerodynamic characteristics as  $M_{\dot{\alpha}}$ , only it is in a non-dimensional form.
3. Most aircraft designers and builders are accustomed to speaking in terms of stability derivatives. Therefore, it is a good idea to develop a "feel" for all of the important ones.
4. These equations as well as the parametric equations describe the complete motion of an aircraft. They can be programmed directly into a computer and connected to a flight simulator. They may also be used in cursory design analyses. Due to their simplicity, they are especially useful as an analytical tool to investigate aircraft handling qualities and determine the effects of changes in aircraft design.

## Basic Force Relations

To aid in developing the stability derivatives from the basic force relations, the following table is provided.

LONGITUDINAL MOTION			
Equation	Coefficient	Normalizing Factor	Parameters
Drag	$D = \frac{1}{2} \rho U^2 S C_D$	$\frac{1}{m}$	$D_u$ ( $D_\alpha$ requires special derivation) ( $D_{\dot{e}}$ , $D_{\dot{i}}$ and $D_q$ are insignificant)
Lift	$L = \frac{1}{2} \rho U^2 S C_L$	$\frac{1}{mU_0}$	$L_u$ $L_i$ $L_q$ $L_{\dot{e}}$ $L_{\dot{i}}$
Pitch	$M = \frac{1}{2} \rho U^2 S c C_m$	$\frac{1}{I_y}$	$m_u$ $m_i$ $m_q$ $m_{\dot{e}}$ $m_{\dot{i}}$
Independent Variables	Linear Velocity	Angles	Angular Rates
	$u$	$\alpha$ $\epsilon$	$\dot{\alpha}$ $q$
Compensating Factors	$\frac{1}{U_0}$	None	$\frac{c}{2U_0}$

LATERAL-DIRECTIONAL MOTION			
Equation	Coefficient	Normalizing Factor	Parameters
Side	$Y = \frac{1}{2} \rho U^2 S C_Y$	$\frac{1}{mU_0}$	$Y_\beta$ $Y_p$ $Y_r$ $Y_{\dot{\beta}}$ $Y_{\dot{r}}$ ( $Y_{\dot{\beta}}$ and $Y_{\dot{r}}$ are insignificant)
Roll	$L = \frac{1}{2} \rho U^2 S b C_L$	$\frac{1}{I_x}$	$L_\beta$ $L_p$ $L_r$ $L_{\dot{\beta}}$ $L_{\dot{r}}$ ( $L_{\dot{\beta}}$ is insignificant)
Yaw	$N = \frac{1}{2} \rho U^2 S b C_N$	$\frac{1}{I_z}$	$N_\beta$ $N_p$ $N_r$ $N_{\dot{\beta}}$ $N_{\dot{r}}$ ( $N_{\dot{\beta}}$ is insignificant)
	Independent Variables	Angles	Angular Rates
		$\beta$ $\phi$ $\psi$	$\dot{\beta}$ $p$ $r$
Compensating Factors		None	$\frac{b}{2U_0}$

## Non-Dimensional Derivative Equations

A list of the non-dimensional derivative equations is presented below.

### NON-DIMENSIONAL LONGITUDINAL EQUATIONS

	( $\theta$ )	( $\hat{u}$ )	( $\alpha$ )	
Drag	$C_{L_o} \theta$	$(2u_D + C_{D_u} + 2C_{D_0}) \hat{u}$	$C_{D_\alpha} \alpha$	$= 0$
Lift	$(2u_D - C_{L_q} D) \theta$	$(C_{L_u} + 2C_{L_o}) \hat{u}$	$-(2u_D + C_{L_\alpha} D + C_{L_\alpha}) \alpha$	$= C_{L_{\delta e}} \delta_e$
Pitching Moment	$(i_Y D^2 - C_{m_q} D) \theta$	$- C_{m_u} \hat{u}$	$-(C_{m_\alpha} D + C_{m_\alpha}) \alpha$	$= C_{m_{\delta e}} \delta_e$

$$i_Y = \frac{8I_Y}{\rho S c^3} \quad u = \frac{2m}{\rho S c}$$

$$D = \frac{c}{2U_o} \frac{d(\quad)}{dt}$$

### NON-DIMENSIONAL LATERAL-DIRECTIONAL EQUATIONS

	( $\beta$ )	( $\phi$ )	( $\hat{r}$ )	
Side Force	$(2u_D - C_{Y_\beta}) \beta$	$-(C_{Y_p} D + C_{L_o}) \phi$	$+(2u_D - C_{Y_r}) \hat{r}$	$= C_{Y_{\delta r}} \delta_r$
Rolling Moment	$- C_{l_\beta} \beta$	$(i_x D^2 - C_{l_p} D) \phi$	$-(i_{xz} D + C_{l_r}) \hat{r}$	$= C_{l_{\delta a}} \delta_a + C_{l_{\delta r}} \delta_r$
Yawing Moment	$- C_{n_\beta} \beta$	$-(i_{xz} D^2 + C_{n_p} D) \phi$	$+(i_z D - C_{n_r}) \hat{r}$	$= C_{n_{\delta a}} \delta_a + C_{n_{\delta r}} \delta_r$

$$i_x = \frac{8I_x}{\rho S b^3} \quad D = \frac{b}{2U_o} \frac{d(\quad)}{dt}$$

$$i_{xz} = \frac{8I_{xz}}{\rho S b^3} \quad u = \frac{2m}{\rho S b}$$

$$i_z = \frac{8I_z}{\rho S b^3}$$

# LONGITUDINAL STATIC STABILITY

## 3.1 DEFINITION OF LONGITUDINAL STATIC STABILITY

Static stability is the reaction of a body to a disturbance from equilibrium. To determine the static stability of a body, the body must be initially disturbed from its equilibrium state. If when disturbed from equilibrium, the body returns to its original equilibrium position, the body displays positive static stability or is stable. If the body remains in the disturbed position, the body is said to be neutrally stable. However, should the body, when disturbed, continue to displace from equilibrium, the body has negative static stability or is unstable.

Longitudinal static stability or "gust stability" of an aircraft is determined similarly. If an aircraft in equilibrium is momentarily disturbed by a vertical gust, the resulting change in angle of attack causes changes in lift coefficients on the aircraft. (Velocity is constant for this time period.) The changes in lift coefficients produce additional aerodynamic forces and moments in this disturbed position. If the aerodynamic forces and moments created tend to return the aircraft to its original undisturbed condition, the aircraft possesses positive static stability or is stable. Should the aircraft remain in the disturbed position, it possesses neutral stability. If the forces and moments cause the aircraft to diverge further from equilibrium, the aircraft possesses negative longitudinal static stability or is unstable.

## 3.2 ANALYSIS OF LONGITUDINAL STATIC STABILITY

Longitudinal static stability is only a special case for the total equations of motion of an aircraft. Of the six equations of motion, longitudinal static stability is concerned with only one, the pitch equation, that equation describing the aircraft's motion about the y - axis.

$$G_y = \dot{q}I_y - p r(I_z - I_x) + (p^2 - r^2)I_{xz} \quad (3.1)$$

The fact that theory pertains to an aircraft in straight, steady, symmetrical flight with no unbalance of forces or moments permits longitudinal static stability motion to be independent of the lateral and directional equations of motion. This is not an oversimplification since most aircraft spend much of the flight under symmetric equilibrium conditions. Furthermore the disturbance required for stability determination and the measure of the aircraft's response takes place about the y - axis or in the longitudinal plane.

Since longitudinal static stability is concerned with resultant aircraft pitching moments caused by momentary changes in angle of attack and lift coefficients, the primary stability derivatives become  $C_{m_{\dot{\alpha}}}$  and  $C_{m_{\dot{\alpha}_L}}$ . The value of either derivative is a direct indication of the longitudinal static stability of the particular aircraft.

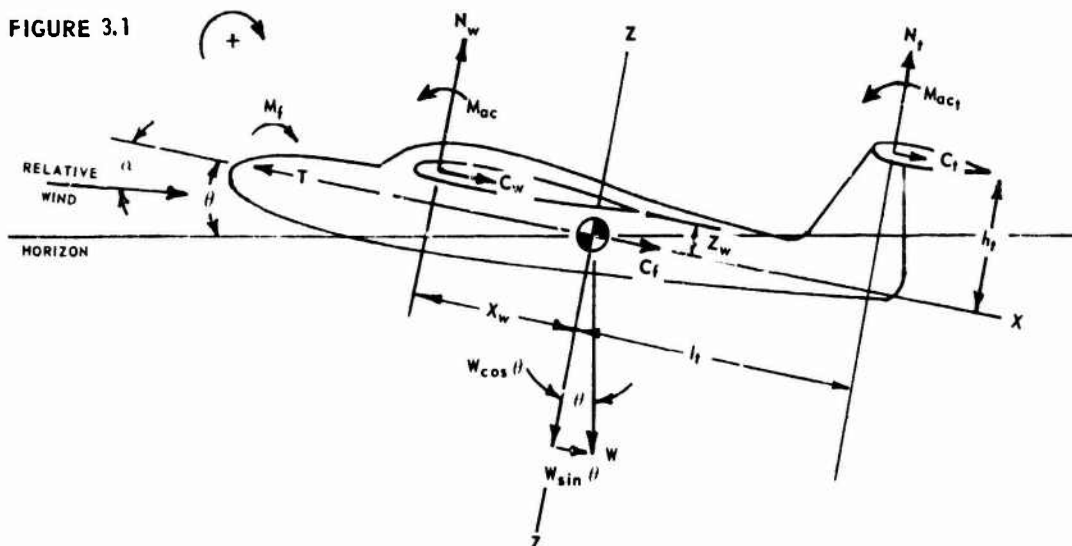
To determine an expression for the derivative,  $C_{m_{\dot{\alpha}}}$ , an air-

craft in stabilized, equilibrium flight with horizontal stabilizer control surface fixed will be analyzed. A moment equation will be determined from the forces and moments acting on the aircraft. Once this equation is nondimensionalized, in moment coefficient form, the derivative with respect to  $C_L$  will be taken. This differential equation will be an expression for  $C_{m_{C_L}}$  and will relate directly to the aircraft's stability. Individual term contribution to stability will in turn be analyzed. A flight test relationship for determining the stability of an aircraft will be developed followed by analysis of the aircraft with a free control surface.

### ● 3.3 THE STABILITY EQUATION

To derive the longitudinal pitching moment equation, refer to the aircraft in figure 3.1. Writing the moment equation using the sign convention of pitchup being a positive moment and assuming a small angle of attack  $\alpha$  so that  $\cos \alpha \approx 1$  and  $\sin \alpha \approx \alpha$ ;

FIGURE 3.1



$$M_{CG} = N_w X_w + C_w Z_w - M_{ac} + N_f - N_T \ell_T + C_T h_T - M_{ac_T} \quad (3.2)$$

If an order of magnitude check is made, some of the terms can be logically eliminated because of their relative size.  $C_T$  can be omitted since

$$C_T = \frac{C_w}{10} = \frac{N_w}{100}$$

$M_{ac_T}$  is zero for a symmetrical airfoil horizontal stabilizer section. Rewriting the simplified equation:

$$M_{CG} = N_w X_w + C_w Z_w - M_{ac} + M_f - N_T \ell_T \quad (3.3)$$

It is convenient to express equation 3.3 in nondimensional coefficient form by dividing both sides of the equation by  $q_w S_w c_w$

$$\frac{M_{CG}}{q_w S_w c_w} = \frac{N_w X_w}{q_w S_w c_w} + \frac{C_w Z_w}{q_w S_w c_w} - \frac{M_{ac}}{q_w S_w c_w} + \frac{M_f}{q_w S_w c_w} - \frac{N_T \ell_T}{q_w S_w c_w} \quad (3.4)$$

Substituting the following coefficients in equation 3.4:

$$C_{m_{CG}} = \frac{M_{CG}}{q_w S_w c_w} \quad \text{total pitching moment coefficient}$$

$$C_{m_{ac}} = \frac{M_{ac}}{q_w S_w c_w} \quad \text{wing aerodynamic pitching moment coefficient}$$

$$C_{m_f} = \frac{M_f}{q_w S_w c_w} \quad \text{fuselage aerodynamic pitching moment coefficient}$$

$$C_N = \frac{N_w}{q_w S_w} \quad \text{wing aerodynamic normal force coefficient}$$

$$C_{N_T} = \frac{N_T}{q_T S_T} \quad \text{tail aerodynamic normal force coefficient}$$

$$C_c = \frac{C_w}{q_w S_w} \quad \text{wing aerodynamic chordwise force coefficient}$$

Equation 3.4 may now be written:

$$C_{m_{CG}} = C_N \frac{x_w}{c} + C_c \frac{z_w}{c} - C_{m_{ac}} + C_{m_f} - \frac{N_T \ell_T}{q_w S_w c_w} \quad (3.5)$$

where  $C$  and  $c_w$  are used interchangeably to represent the mean aerodynamic chord of the wing. To have the tail term in terms of a coefficient, multiply and divide the term by  $q_T S_T$

$$\frac{N_T \ell_T}{q_w S_w c_w} = \frac{q_T S_T}{q_T S_T} \cdot \frac{N_T \ell_T}{q_T S_T}$$

Substituting tail efficiency factor  $\eta_T = q_T/q_w$  and designating tail volume coefficient  $V_H = \ell_T S_T / c_w S_w$ , Equation (3.5) becomes:

$$C_{m_{CG}} = C_N \frac{x_w}{c} + C_c \frac{z_w}{c} - C_{m_{ac}} + C_{m_f} - C_{N_T} V_H \eta_T \quad (3.6)$$

Equation 3.6 is referred to as the equilibrium equation in pitch. If the magnitudes of the individual terms in the above equation are adjusted to the proper value, the aircraft may be placed in equilibrium flight where  $C_{m_{CG}} = 0$ .

Taking the derivative of equation 3.6 with respect to  $C_L$  and assuming that  $x_w$ ,  $z_w$ ,  $V_H$ , and  $\eta_T$  do not vary with  $C_L$ ,

$$\begin{aligned} \frac{dC_{m_{CG}}}{dC_L} &= \underbrace{\frac{dC_N}{dC_L} \frac{x_w}{c} + \frac{dC_c}{dC_L} \frac{z_w}{c} - \frac{dC_{m_{ac}}}{dC_L}}_{\text{WING}} \\ &+ \underbrace{\frac{dC_{m_f}}{dC_L}}_{\text{FUSELAGE}} - \underbrace{\frac{dC_{N_T}}{dC_L} V_H \eta_T}_{\text{TAIL}} \quad (3.7) \end{aligned}$$

Equation 3.7 is the stability equation and is related to the stability derivative  $C_{m_\alpha}$  by the slope of the lift curve,  $a$ .

$$\frac{dC_m}{d\alpha} = a \frac{dC_m}{dC_L} \quad (3.8)$$

Equation 3.6 and equation 3.7 determine the two criteria necessary for longitudinal stability:

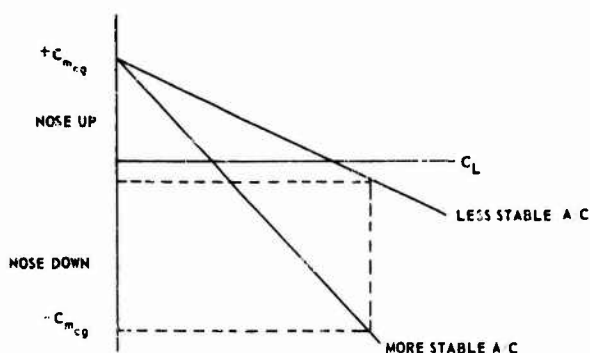
1. The aircraft is balanced.
2. The aircraft is stable.

The final condition is satisfied if the pitching moment equation may be forced to  $C_{m_{CG}} = 0$  for all useful positive values of  $C_L$ . This condition is achieved by

trimming the aircraft so that moments about the center of gravity are zero (i.e.,  $M_{CG} = 0$ ).

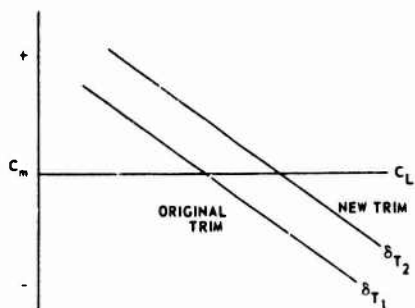
The second condition is satisfied if equation 3.7 or  $dC_{mCG}/dC_L$  has a negative value. From figure 3.2 a negative value for equation 3.7 is necessary if the aircraft is to be stable. Should a gust cause an angle of attack increase and a corresponding increase in  $C_L$ , a negative  $C_{mCG}$  should be produced to return the aircraft to equilibrium, or  $C_{mCG} = 0$ . The greater the slope or the negative value, the more restoring moment is generated for an increase in  $C_L$ . The slope or  $dC_m/dC_L$  is a direct measure of the "gust stability" of the aircraft.

FIGURE 3.2



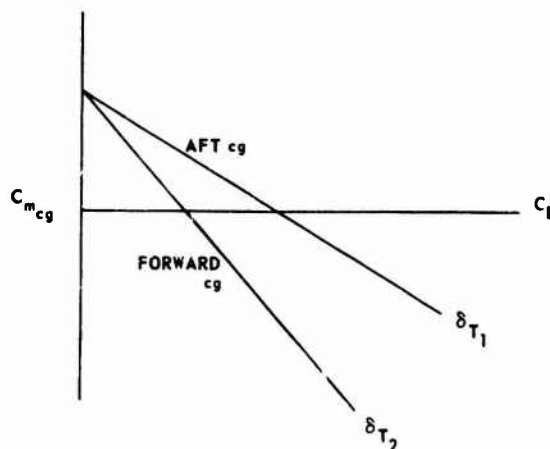
If the aircraft is retrimmed from one angle of attack to another, the basic stability of the aircraft or slope  $dC_m/dC_L$  does not change. Note figure 3.3.

FIGURE 3.3



However, if the cg is changed or values of  $X_W$ ,  $Z_W$ , and  $V_H$  are changed, the slope or stability of the aircraft is changed. See equation 3.7. For no change in trim tab setting, the stability curve may shift as in figure 3.4.

FIGURE 3.4

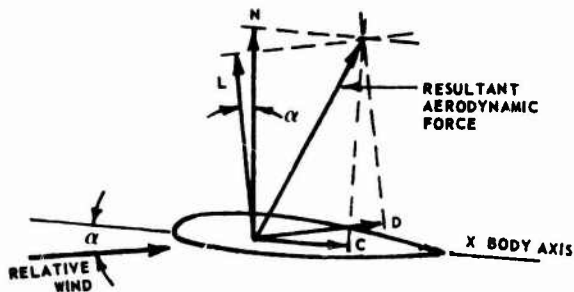


### 3.4 EXAMINATION OF THE WING, FUSELAGE, AND TAIL CONTRIBUTION TO THE STABILITY EQUATION

#### The Wing Contribution to Stability:

The lift and drag are by definition always perpendicular and parallel to the relative wind. It is therefore inconvenient to use these forces to obtain moments, for their arms to the center of gravity vary with angle of attack. For this reason, all forces are resolved into normal and chordwise forces whose axes remain fixed with the aircraft and whose arms are therefore constant:

FIGURE 3.5



Assuming the wing lift to be the airplane lift and the angle of attack of the wing to be the airplane's angle of attack, the following relationship exists between the normal and lift forces (figure 3.5):

$$N = L \cos \alpha + D \sin \alpha \quad (3.9)$$

$$C = D \cos \alpha - L \sin \alpha \quad (3.10)$$

Therefore, the coefficients are similarly related:

$$C_N = C_L \cos \alpha + C_D \sin \alpha \quad (3.11)$$

$$C_C = C_D \cos \alpha - C_L \sin \alpha \quad (3.12)$$

The stability contributions,  $dC_N/dC_L$  and  $dC_C/dC_L$ , are obtained:

$$\begin{aligned} \frac{dC_N}{dC_L} &= \frac{dC_L}{dC_L} \cos \alpha - C_L \frac{d\alpha}{dC_L} \sin \alpha \\ &\quad + \frac{dC_D}{dC_L} \sin \alpha + C_D \frac{d\alpha}{dC_L} \cos \alpha \end{aligned} \quad (3.13)$$

$$\begin{aligned} \frac{dC_C}{dC_L} &= \frac{dC_D}{dC_L} \cos \alpha - C_D \frac{d\alpha}{dC_L} \sin \alpha \\ &\quad - \frac{dC_L}{dC_L} \sin \alpha - C_L \frac{d\alpha}{dC_L} \cos \alpha \end{aligned} \quad (3.14)$$

Making an additional assumption that:

$$C_T = C_{D_{\text{parasite}}} + \frac{C_L^2}{\pi A R e} \quad \text{and if}$$

$C_{D_{\text{parasite}}}$  is constant with change in  $C_L$ :

$$\text{Then } \frac{dC_D}{dC_L} = \frac{2C_L}{\pi A R e}$$

If the angles of attack are small such that  $\cos \alpha = 1.0$  and  $\sin \alpha \approx \alpha$ , equations 3.13 and 3.14 become:

$$\begin{aligned} \frac{dC_N}{dC_L} &= 1 + C_L \alpha \left( \frac{2}{\pi A R e} - \frac{d\alpha}{dC_L} \right) \\ &\quad + C_D \frac{d\alpha}{dC_L} \end{aligned} \quad (3.15)$$

$$\begin{aligned} \frac{dC_C}{dC_L} &= \frac{2}{\pi A R e} C_L - C_D \frac{d\alpha}{dC_L} \alpha \\ &\quad - \alpha - C_L \frac{d\alpha}{dC_L} \end{aligned} \quad (3.16)$$

Examining the above equations for relative magnitude,

$C_D$  is on the order of 0.3

$C_L$  usually ranges from .2 to 2.0

$\alpha$  is small,  $\approx .2$  radians

$\frac{d\alpha}{dC_L}$  is nearly constant at .2 radians

$\frac{2}{\pi A R e}$  is on the order of .1

Making these substitutions, equations 3.15 and 3.16 become

$$\begin{aligned} \frac{dC_N}{dC_L} &= 1 - .04 + .06 \\ &= 1.02 \approx 1.0 \end{aligned} \quad (3.17)$$

$$\begin{aligned}\frac{dC_C}{dC_L} &= .1 C_L - .012 - .2 - .2C_L \\ &= - (.2 + .1 C_L)\end{aligned}\quad (3.18)$$

By definition the coefficient of moment about the aerodynamic center is invariant with respect to angle of attack. Therefore,

$$\frac{dC_{mac}}{dC_L} = 0$$

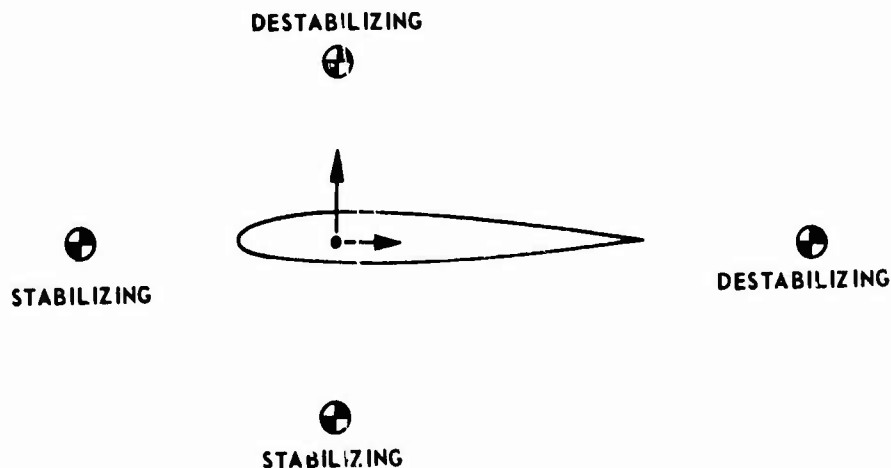
Rewriting the wing contribution of the stability equation 3.7,

$$\frac{dC_m}{dC_{L_{WING}}} = \frac{x_w}{c} - (.2 + .1 C_L) \frac{z_w}{c} \quad (3.19)$$

From figure 3.5 when  $\alpha$  increases, the normal force increases and the chordwise force decreases. Equation 3.19 shows the relative magnitude of these changes. The position of the cg above or below the aircraft chord (a.c.) has a much smaller effect on stability than does the position of the cg ahead or behind the a.c. With the cg ahead of the a.c., the normal force is stabilizing. From equation 3.19, the more forward the cg location, the more stable the aircraft. With the cg below the a.c., the chordwise force is stabilizing since this force decreases as the angle of attack increases. The further the cg is located below the a.c., the more stable the aircraft or the more negative the value of  $dC_m/dC_L$ . The wing contribution to stability depends on the cg and a.c. relationship shown in figure 3.6.

FIGURE 3.6

WING CONSTRUCTION TO STABILITY



For a stable wing contribution to stability, the aircraft should be designed with a high wing aft of the center of gravity.

### Symmetrical Wing Contribution.

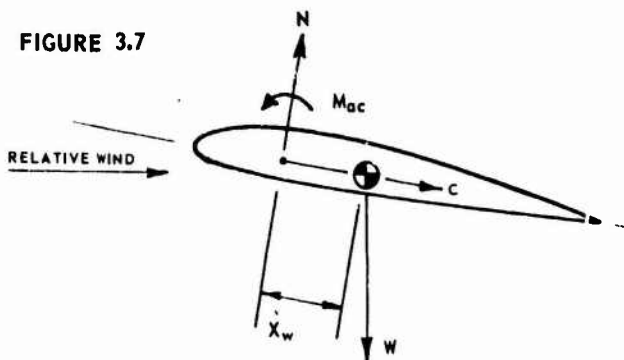
Fighter type aircraft and most low wing, large aircraft have cg's very close to the top of the mean aerodynamic chord.  $z_w$  is on the order of .03. For these aircraft, the chordwise force contribution to stability can be neglected. The wing contribution then becomes:

$$\frac{dC_m}{dC_{L_{WING}}} = \frac{x_w}{c} \quad (3.20)$$

### The Flying Wing.

In order for a flying wing to be a usable aircraft, it must be balanced (fly in equilibrium at a useful positive  $C_L$ ) and be stable. The problem may be analyzed as follows:

FIGURE 3.7



For the wing in figure 3.7, assuming that the chordwise force acts through the cg, the equilibrium equation in pitch may be written:

$$M_{CG} = NX_w - M_{ac} \quad (3.21)$$

Writing the equation in coefficient form,

$$C_{m_{CG}} = C_N \frac{x_w}{c} - C_{m_{ac}} \quad (3.22)$$

For controls fixed, the stability equation becomes,

$$\frac{dC_{m_{CG}}}{dC_L} = \frac{dC_N}{dC_L} \frac{x_w}{c} \quad (3.23)$$

Equations 3.22 and 3.23 show that the wing in figure 3.7, is balanced and unstable. To make the wing stable, or  $dC_m/dC_L$  negative, the center of gravity must be ahead of the wing aerodynamic center. Making this cg change, however, now changes the signs in equation 3.21. The equilibrium and stability equations become:

$$C_{m_{CG}} = -C_N \frac{x_w}{c} - C_{m_{ac}} \quad (3.24)$$

$$\frac{dC_{m_{CG}}}{dC_L} = -\frac{dC_N}{dC_L} \frac{x_w}{c} \quad (3.25)$$

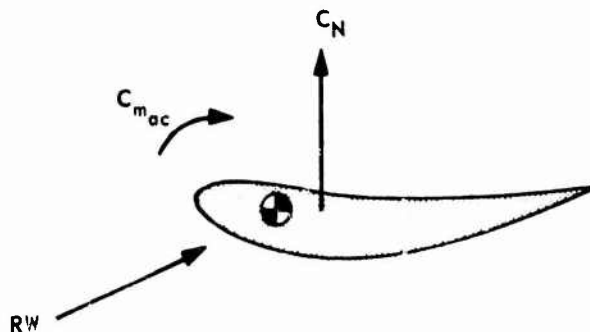
The wing is now stable but unbalanced. The balanced condition is possible with a positive  $C_{m_{ac}}$ .

Three methods of obtaining a positive  $C_{m_{ac}}$  are:

1. Use a negative camber airfoil section. The positive  $C_{m_{ac}}$  will give a flying wing that is stable and balanced (figure 3.8).

FIGURE 3.8

### NEGATIVE CAMBERED FLYING WING



This type of wing is not realistic because of unsatisfactory dynamic characteristics, small  $c_g$  range, and extremely low  $C_L$  capability.

2. A reflexed airfoil section reduces the effect of camber by creating a download near the trailing edge. Similar results are possible with an upward deflected flap on a symmetrical airfoil.
3. A symmetrical airfoil section in combination with sweep and wing tip washout (reduction in angle of incidence at the tip) will produce a positive  $C_{mac}$  by virtue of the aerodynamic couple produced between the down loaded tips and the normal lifting force.

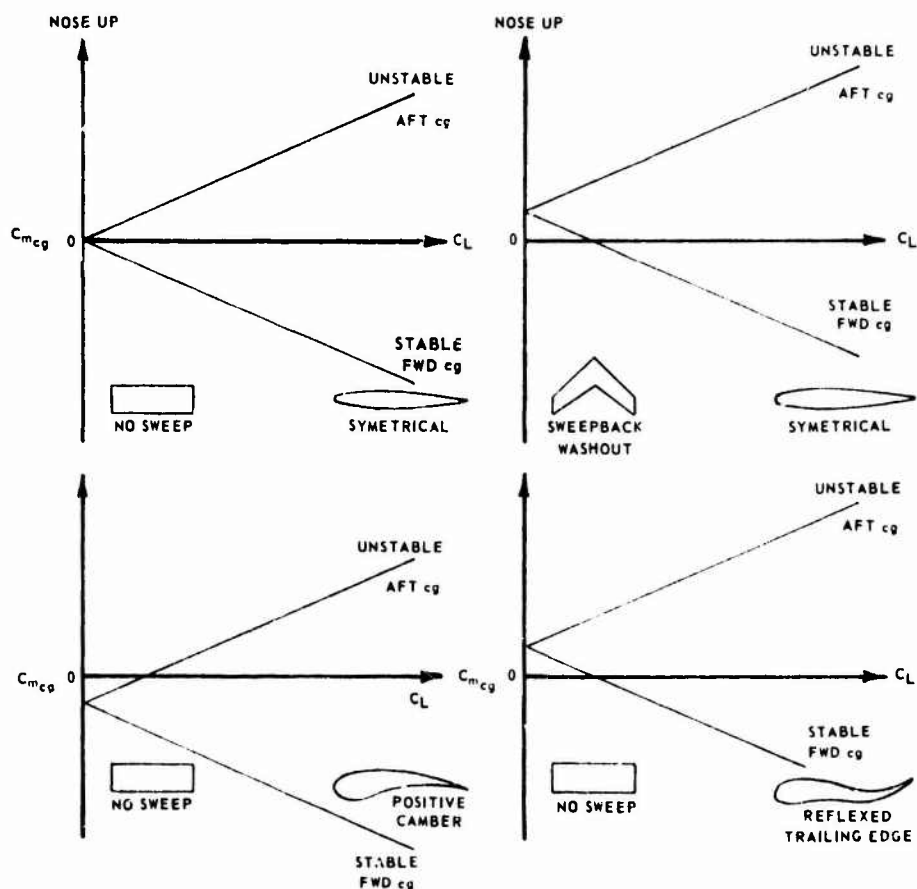
Figure 3.9 shows idealized  $C_{m_{cg}}$  versus  $C_L$  for various wings in a control fixed position. Only two of the wings are capable of sustained flight.

#### The Fuselage Contribution to Stability:

The fuselage contribution is difficult to separate from the wing terms because it is strongly influenced by interference from the wing flow field. Wind tunnel tests of the wing body combination are used by airplane designers to obtain information about the fuselage influence on stability.

A fuselage by itself is almost always destabilizing because the center of pressure is usually

FIGURE 3.9



ahead of the center of gravity. The magnitude of the destabilizing effects of the fuselage requires their consideration in the equilibrium and stability equations.

$$\frac{dC_m}{dC_{L_{Fus}}} = \text{Positive quantity}$$

#### The Tail Contribution to Stability:

From equation 3.7, the tail contribution to stability was found to be:

$$\frac{dC_m}{dC_{L_{Tail}}} = - \frac{dC_{N_T}}{dC_L} V_H \eta_T \quad (3.26)$$

For small angles of attack, the lift curve slope of the tail is very nearly the same as the slope of the normal force curve (i.e.,  $C_{N_T} \approx C_{L_T}$ ).

$$a_T = \frac{dC_L}{d\alpha_{Tail}} \approx \frac{dC_N}{d\alpha_{Tail}} \quad (3.27)$$

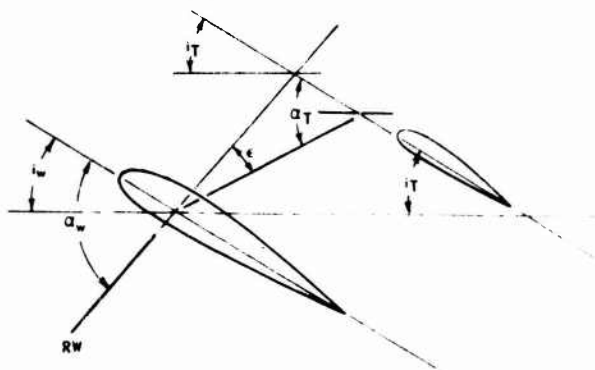
Therefore:

$$C_{N_T} = a_T \alpha_T \quad (3.28)$$

An expression for  $\alpha_T$  in terms of  $C_L$  is required before solving for  $dC_{N_T}/dC_L$ .

FIGURE 3.10

TAIL ANGLE OF ATTACK



From figure 3.10,

$$\alpha_T = \alpha_w - i_w + i_T - \epsilon \quad (3.29)$$

Substituting equation 3.29 into 3.28 and taking the derivative with respect to  $C_L$ , where  $a_w = dC_L/d\alpha$

$$\begin{aligned} \frac{dC_{N_T}}{dC_L} &= a_T \left( \frac{d\alpha_w}{dC_L} - \frac{d\epsilon}{dC_L} \right) \\ &= a_T \left( \frac{1}{a_w} - \frac{d\epsilon}{d\alpha} \frac{1}{a_w} \right) \end{aligned} \quad (3.30)$$

upon factoring out  $1/a_w$ ,

$$\frac{dC_{N_T}}{dC_L} = \frac{a_T}{a_w} \left( 1 - \frac{d\epsilon}{d\alpha} \right) \quad (3.31)$$

Substituting equation 3.31 into 3.26, the expression for the tail contribution becomes,

$$\frac{dC_m}{dC_{L_{Tail}}} = - \frac{a_T}{a_w} \left( 1 - \frac{d\epsilon}{d\alpha} \right) V_H \eta_T \quad (3.32)$$

The value of  $a_T/a_w$  is very nearly constant. These values are usually obtained from experimental data.

The tail volume coefficient,  $V_H$ , is a term determined by the geometry of the aircraft. To vary this term is to redesign the aircraft.

$$V_H = \frac{l_T S_T}{c S} \quad (3.33)$$

The further the tail is located aft of the cg (increase  $l_T$ ) or the greater the tail surface area ( $S_T$ ), the greater the tail volume coefficient  $V_H$  which increases the tail contribution to stability.

The expression,  $\eta_T$ , is the ratio of the tail dynamic pressure to the wing dynamic pressure and  $\eta_T$  is greater than unity for a prop aircraft and less than unity for a jet. For power-off considerations,  $\eta_T \approx 1.0$ .

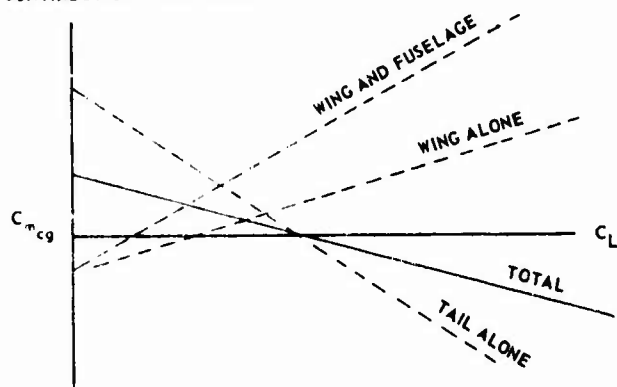
The term  $(1 - d\epsilon/d\alpha)$  is an important factor in the stability contribution of the tail. Large positive values of  $d\epsilon/d\alpha$  produce destabilizing effects by reversing the sign of the term  $(1 - d\epsilon/d\alpha)$  and consequently, the sign of  $dC_m/dC_{L_{Tail}}$ .

For example at high angles of attack the F-104 experiences a sudden increase in  $d\epsilon/d\alpha$ . The term  $(1 - d\epsilon/d\alpha)$  goes negative causing the entire tail contribution to be positive or destabilizing, causing aircraft pitchup. The stability of an aircraft is definitely influenced by the wing vortex system. For this reason the downwash variation with angle of attack should be evaluated in the wind tunnel.

The horizontal stabilizer provides the necessary positive stability contribution (negative  $dC_m/dC_L$ ) to offset the negative stability of the wing and fuselage combination and to make the entire aircraft stable and balanced (figure 3.11).

FIGURE 3.11

CONTRIBUTIONS TO STABILITY



The stability equation 3.7 may now be written as,

$$\frac{dC_m}{dC_L} = \frac{x_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) \quad (3.33a)$$

### The Power Contribution to Stability:

The addition of a power plant to the aircraft may have a decided effect on the equilibrium as well as the stability equations. The overall effect may be quite complicated. This section will be a qualitative discussion of the power effects. The actual end result as to the power effects on trim and stability should come from large scale wind tunnel models or actual flight test.

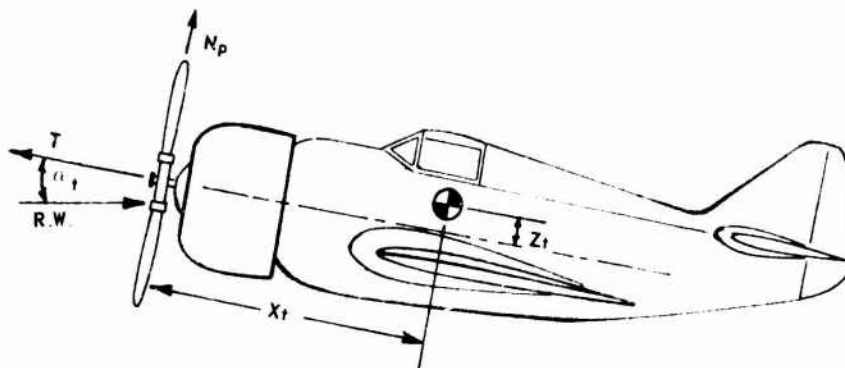
The power effects on a propeller-driven aircraft which influence the static longitudinal stability of the aircraft are:

1. Thrust force effect - effect on stability from the thrust force acting along the propeller axis.
2. Normal force effect - effect on stability from a force normal to the thrust line and in the plane of the propeller.
3. Indirect effects - power plant effects on the stability contribution of other parts of the aircraft.



FIGURE 3.12

PROPELLER THRUST AND NORMAL FORCE



Writing the moment equation for the power terms as:

$$M_{CG} = TZ_T + N_p X_T \quad (3.34)$$

In coefficient form,

$$C_{m_{CG}} = C_T \frac{Z_T}{c} + C_{N_p} \frac{X_T}{c} \quad (3.35)$$

The direct power effect on the aircraft's stability equation is then:

$$\frac{dC_{m_{CG}}}{dC_L} = \frac{dC_T}{dC_L} \frac{Z_T}{c} + \frac{dC_{N_p}}{dC_L} \frac{X_T}{c} \quad (3.36)$$

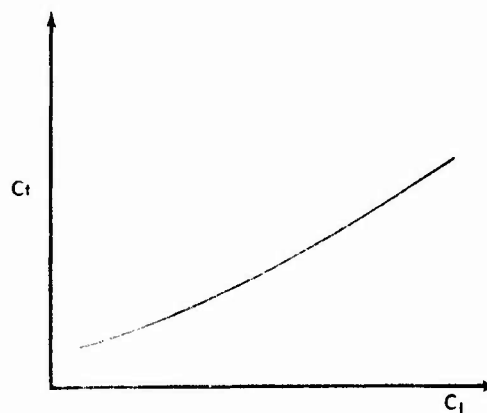
The sign of  $dC_{m_{CG}}/dC_L$  then depends on the sign of the derivatives  $dC_{N_p}/dC_L$  and  $dC_T/dC_L$ .

We shall first consider the  $dC_T/dC_L$  derivative. As speed varies at different flight conditions, throttle position is held constant. Consequently,  $C_T$  varies in a manner that can be represented by  $dC_T/dC_L$ . The coefficient of thrust for a reciprocating power plant varies with  $C_L$  and propeller efficiency. Propeller efficiency which is avail-

able from propeller performance estimates in manufacturer's data, decreases with increase in velocity. Coefficient of thrust variation with  $C_L$  is nonlinear with the derivative large at low speeds. The combination of these two variations approximately linearize  $C_T$  versus  $C_L$  (figure 3.13). The sign of  $dC_{m_{CG}}/dC_L$  is positive.

FIGURE 3.13

COEFFICIENT OF THRUST CURVE RECIPROCATING POWER PLANT WITH PROPELLER



The derivative,  $dC_{N_p}/dC_L$ , is positive since the normal propeller force increases linearly with the local angle of attack of the propeller axis,  $\alpha_p$ .

The direct power effects are then destabilizing if the cg is as

shown in figure 3.12, or where the power plant is ahead and below the cg.

$$\frac{dC_m}{dC_{L_{power}}} = \frac{dC_T}{dC_L} \frac{Z_T}{c} + \frac{dC_{N_T}}{dC_L} \frac{X_T}{c} \quad (3.37)$$

The indirect power effects must also be considered in evaluating the overall stability contribution of the propeller power plant. No attempt will be made to determine their quantitative magnitudes; however, their general influence on the aircraft's stability and trim condition can be great.

#### 1. Increase of Angle of Downwash, $\epsilon$ :

Since the normal force on the propeller increases with angle of attack under powered flight, the slipstream is deflected downward netting an increase downwash at the tail. The downwash in the slipstream will increase more rapidly with angle of attack than the downwash outside the slipstream. The derivative  $d\epsilon/d\alpha$  has a positive increase with power. The term  $(1 - d\epsilon/d\alpha)$  in equation 3.32 is reduced causing the tail trim contribution to be less negative or less stable than the power-off situation.

#### 2. Increase of $\eta_T = (q_T/q_w)$ :

The dynamic pressure,  $q_T$ , of the tail is increased by the slipstream and  $\eta_T$  is greater than unity. From equation 3.32, the increase of  $\eta_T$  with addition of a power plant increases the tail contribution to stability. However, if the tail is carrying a download at trim and if it should move into a high velocity region of the slipstream at higher  $C_L$ , more of a noseup moment would be present as  $C_L$  increased, causing an obvious destabilizing effect.

Both slipstream effects mentioned above may be reduced by lo-

cating the horizontal stabilizer high on the tail and out of the slipstream at operating angles of attack.

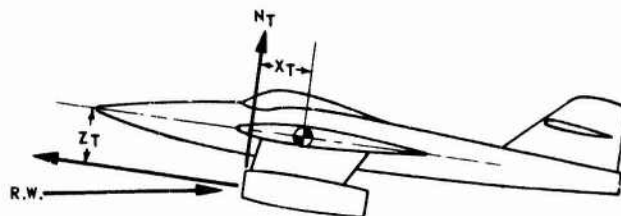
#### Power Effects on Jet Aircraft.

The magnitude of the power effects on jet-powered aircraft are generally smaller than on propeller-driven aircraft. By assuming that jet engine thrust does not change with velocity or angle of attack, and by assuming constant power settings, smaller power effects would be expected than with a similar reciprocating engine aircraft.

There are three major contributions of a jet engine to the equilibrium static longitudinal stability of the aircraft. These are the direct thrust effects, the normal force effects at the air duct inlet, and the indirect effect of the induced flow at the tail.

The thrust and normal force contribution may be determined from figure 3.13a.

FIGURE 3.13a  
TEST THRUST AND NORMAL FORCE



Writing the equation,

$$M_{CG} = T Z_T + N_T X_T \quad (3.38)$$

or

$$C_{m_{CG}} = \frac{T}{qSc} Z_T + C_{N_T} \frac{X_T}{c} \quad (3.39)$$

With the aircraft in unaccelerated flight, the dynamic pressure is a function of lift coefficient.

$$q = \frac{W}{C_L S} \quad (3.40)$$

Therefore,

$$C_{m_{CG}} = \frac{T}{W} \frac{z_T}{c} C_L + C_{N_T} \frac{x_T}{c} \quad (3.41)$$

If thrust is considered independent of speed,\* then

$$\frac{dC_{m_{CG}}}{dC_L} = \frac{T}{W} \frac{z_T}{c} + \frac{dC_{N_T}}{dC_L} \frac{x_T}{c} \quad (3.42)$$

The thrust contribution to stability then depends on whether the thrust line is above or below the cg. Locating the engine below the cg causes a destabilizing influence, and above the cg a stabilizing influence.

The normal force contribution depends on the sign of the derivative  $dC_{N_T}/dC_L$ . The normal force  $N_T$  is created at the air-duct inlet to the turbojet unit. This force is created as a result of the momentum change of the free stream which bends to flow along the duct axis. The magnitude of the force is a function of the mass airflow rate,  $W_a$ , and the angle  $\alpha_T$  between the local flow at the duct entrance and the duct axis.

$$N_T = \frac{W_a}{g} V \alpha_T \quad (3.43)$$

With an increase in  $\alpha_T$ ,  $N_T$  will increase, causing  $dC_{N_T}/dC_L$  to be positive. The normal force contribution will be destabilizing if the inlet duct is ahead of the center of gravity. The magnitude of the destabilizing moment will depend on

the distance the inlet duct is ahead of the center of gravity.

For a jet engine to definitely contribute to positive longitudinal stability, ( $dC_m/dC_L$  negative), the jet engine would be located above and behind the center of gravity.

The indirect contribution of the jet unit to longitudinal stability is the effect of the jet induced downwash at the horizontal tail. This applies to the situation where the jet exhaust passes under or over the horizontal tail surface. The jet exhaust as it discharges from the tail pipe spreads outward. Turbulent mixing causes outer air to be drawn in towards the exhaust area. Downwash at the tail is directly affected. With the exhaust below the tail surface, the downwash is increased, causing the tail term to be less stabilizing.

From the above discussion it can be seen that several factors are important in deciding the power effect on stability. Each aircraft must be examined individually. This is the reason that aircraft are tested for stability in several configurations and at different power settings.

### ● 3.5 THE NEUTRAL POINT

The stick-fixed neutral point is defined as the center of gravity position at which the aircraft displays neutral stability or where  $dC_m/dC_L = 0$ .

The symbol  $h$  is used for center of gravity position where,

$$h = \frac{x_{CG}}{c} \quad (3.44)$$



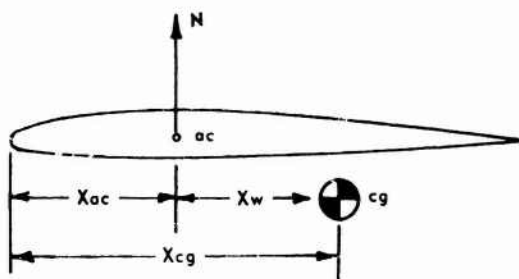
\* For aircraft which have large thrust variation with airspeed, the pitching moment coefficient must be calculated for different values of the aircraft's lift coefficient.

The stability equation for the powerless aircraft is:

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left( 1 - \frac{d\epsilon}{d\alpha} \right) \quad (3.45)$$

Looking at the relationship between cg and a.c. in figure 3.14,

FIGURE 3.14  
cg AND A.C. RELATIONSHIP



$$\frac{X_w}{c} = h - \frac{X_{ac}}{c} \quad (3.46)$$

Substituting equation 3.46 into equation 3.45 and setting  $dC_m/dC_L$  equal to zero,

$$\frac{dC_m}{dC_L} = 0 = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left( 1 - \frac{d\epsilon}{d\alpha} \right) \quad (3.47)$$

Solving for h which is  $h_n$ ,

$$h_n = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_T}{a_w} V_H \eta_T \left( 1 - \frac{d\epsilon}{d\alpha} \right) \quad (3.48)$$

Substituting equation 3.46 back into equation 3.47, the stick-fixed stability derivative in terms of cg positions becomes,

$$\frac{dC_m}{dC_L} = h - h_n \quad (3.49)$$

The stick-fixed static stability is equal to the distance between the cg position and the neutral point in percent of the mean aerodynamic chord. "Static Margin" refers to the same distance but is positive in sign for a stable aircraft.

$$\text{"Static Margin"} = h_n - h \quad (3.50)$$

It is the test pilot's responsibility to evaluate the aircraft's handling qualities and to determine the acceptable static margin for the aircraft.

### 3.6 ELEVATOR POWER

As previously mentioned, for an aircraft to be a usable flying machine, it must possess stability and must be capable of being placed in equilibrium ( $C_{m_{cg}} = 0$ ) throughout the useful  $C_L$  range (balanced).

For trimmed or equilibrium flight,  $C_{m_{cg}}$  must be zero. Some means must be available for balancing the various terms in the moment coefficient of equation 3.51,

$$C_{m_{CG}} = C_N \frac{X_w}{c} + C_C \frac{Z_w}{c} + C_{m_{ac}} + C_{m_f} - a_T \alpha V_H \eta_T \quad (3.51)$$

Several possibilities are available. The center of gravity could be moved fore and aft or up and down thus changing  $X_w/c$  or  $Z_w/c$ . However, this would not only affect the equilibrium lift coefficient but would also change the stability  $dC_m/dC_L$  in the stability equation 3.52. This is undesirable.

$$\frac{dC_m}{dC_L} = \frac{dC_N}{dC_L} \frac{x_w}{c} + \frac{dC_C}{dC_L} \frac{z_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left( 1 - \frac{d\epsilon}{d\alpha} \right) \quad (3.52)$$

The pitching moment coefficient about the aerodynamic center could be changed by effectively changing the camber of the wing by using trailing edge flaps as is done in flying wing vehicles. On the conventional tail-to-the-rear aircraft, trailing edge wing flaps are ineffective in trimming the pitching moment coefficient to zero.

The remaining solution is to change the angle of attack of the horizontal stabilizer to achieve a  $C_{m_{cg}} = 0$  without a change to the basic aircraft stability. The control means is either an elevator on the stabilizer or an all moving stabilizer (called a slab). The slab is used in most high speed aircraft and is the most powerful means of longitudinal control.

Movement of the slab or elevator changes the effective angle of attack of the horizontal stabilizer and, consequently, the lift on the horizontal tail. This in turn changes the moment about the center of gravity due to the horizontal tail. It is of interest to know the amount of pitching moment change associated with a degree of elevator deflection. This may be determined by differentiating equation 3.51 with respect to  $\delta_e$ .

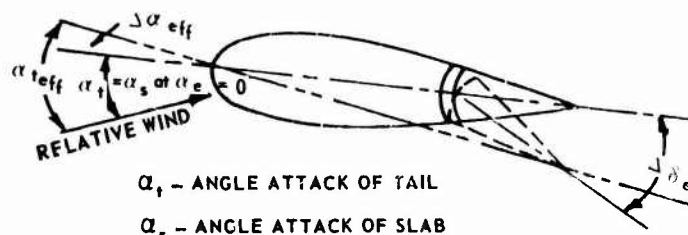
$$\frac{dC_m}{d\delta_e} = C_{m_{\delta_e}} = -a_T V_H \eta_T \frac{d\alpha_T}{d\delta_e} \quad (3.53)$$

This change in pitching moment coefficient with respect to elevator deflection  $C_{m_{\delta_e}}$  is referred to as "elevator power." It indicates the capability of the elevator in producing moments about the center of gravity.

The term  $d\alpha_T/d\delta_e$  in equation 3.53 is termed elevator effectiveness and is given the shorthand notation  $\tau$ . The elevator effectiveness may be considered as the equivalent change in effective tail plane angle of attack per unit change in elevator deflection. The relationship between elevator effectiveness  $\tau$  and the effective angle of attack of the stabilizer is seen in figure 3.15.

FIGURE 3.15

CHANGE IN EFFECTIVE ANGLE OF ATTACK WITH ELEVATOR DEFLECTION



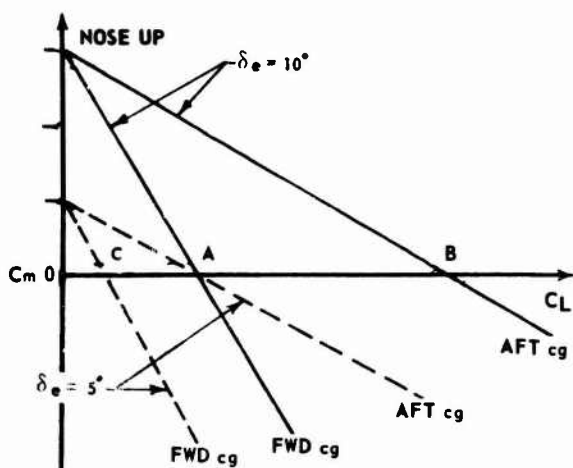
As seen, elevator effectiveness is a design parameter and is determined from wind tunnel tests. Elevator effectiveness is a negative number for all tail to the rear aircraft. The values range from zero to the limiting case of the all moving stabilizer (slab) where  $\tau$  equals  $(-1)$ . The tail angle of attack would change plus one degree for every minus degree the slab moves. For the elevator stabilizer combination, the elevator effectiveness is a function of the ratio of overall elevator area to the entire horizontal tail area.

### ● 3.7 STABILITY CURVES

Figure 3.16 is a wind tunnel plot of  $C_m$  versus  $C_L$  for an aircraft tested under two cg positions and two elevator positions.

FIGURE 3.16

$c_g$  AND  $\delta_e$  VARIATION ON STABILITY



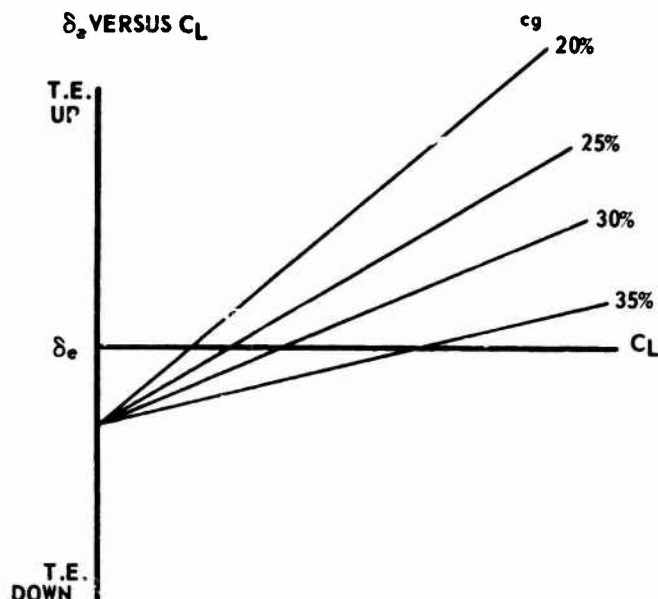
Assuming the elevator effectiveness and the elevator power to be constant, then equal elevator deflections produce equal moments about the  $c_g$ . Points A and B represent the same elevator deflection corresponding to the  $C_{m_{cg}}$  needed to maintain equilibrium. The pilot selects elevator deflection of 10 degrees. In the aft  $c_g$  condition, the aircraft will fly in equilibrium at point B. If the  $c_g$  is moved forward with no change to the elevator deflection, the equilibrium point is now at A or at a new  $C_L$ . Note the increase in the stability of the aircraft (greater negative slope  $dC_m/dC_L$ ).

If the pilot desires to fly at a lower  $C_L$  or at A and not change the  $c_g$ , he does so by deflecting the elevator to 5 degrees. The stability level of the aircraft has not changed (same slope).

A cross plot of figure 3.16 is elevator deflection versus  $C_L$  for  $C_m = 0$ . This is shown in figure 3.17. The slopes of the  $c_g$  curves are indicative of the aircraft's stability.

FIGURE 3.17

$\delta_e$  VERSUS  $C_L$



### 3.8 FLIGHT TEST RELATIONSHIP

The stability equation 3.52 derived previously pertains to theoretical applications and text book solutions. The equation has no use in flight testing. There is no aircraft instrumentation which will measure the change in pitching moment coefficient with change in lift coefficient or angle of attack. Therefore an expression involving parameters easily measurable in flight is required. This expression should relate directly to the stick-fixed longitudinal static stability  $dC_m/dC_L$  of the aircraft.

The external moment acting longitudinally on an aircraft is:

$$M = f(\alpha, \dot{\alpha}, q, V, \delta_e) \quad (3.54)$$

Assuming further that the aircraft is in equilibrium and in unaccelerated flight, then

$$M = f(\alpha, \delta_e) \quad (3.55)$$

Therefore,

$$\Delta M = \frac{\partial M}{\partial \alpha} \Delta \alpha + \frac{\partial M}{\partial \delta_e} \Delta \delta_e \quad (3.56)$$

and

$$C_m = C_{m_\alpha} \alpha + C_{m_{\delta_e}} \delta_e = 0 \quad (3.57)$$

where  $\Delta \alpha = \alpha - \alpha_0 = \alpha$

$$\Delta \delta_e = \delta_e - \delta_{e0} = \delta_e$$

assuming  $\alpha_0 = 0$

$$\delta_{e0} = 0$$

The elevator deflection required to maintain equilibrium is,

$$\delta_e = - \frac{C_{m_\alpha} \alpha}{C_{m_{\delta_e}}} \quad (3.58)$$

Taking the derivative of  $\delta_e$  with respect to  $C_L$ ,

$$\frac{d\delta_e}{dC_L} = - \frac{\frac{dC_{m_\alpha}}{d\alpha} \frac{d\alpha}{dC_L}}{C_{m_{\delta_e}}} = - \frac{\frac{dC_m}{dC_L}}{C_{m_{\delta_e}}} \quad (3.59)$$

In terms of the static margin, the flight test relationship is,

$$\frac{d\delta_e}{dC_L} = \frac{h_n - h}{C_{m_{\delta_e}}} \quad (3.60)$$

The amount of elevator required to fly at equilibrium varies directly as the amount of static stick-fixed stability and inversely as the amount of elevator power.

### 3.9 LIMITATION TO DEGREE OF STABILITY

The degree of stability tolerable in an aircraft is determined by the physical limits of the longitudinal control. The elevator power and amount of elevator deflection is fixed once the aircraft has been designed. If the relationship between  $\delta_e$  required to maintain the aircraft in equilibrium flight and  $C_L$  is linear, then the elevator deflection required to reach any  $C_L$  is,

$$\delta_e = \delta_{e \text{ Zero Lift}} + \frac{d\delta_e}{dC_L} C_L \quad (3.61)$$

The elevator stop determines the absolute limit of the elevator deflection available. Similarly, the elevator must be capable of bringing the aircraft into equilibrium at  $C_{L \text{ Max}}$ .

Recalling the flight test relationship,

$$\frac{d\delta_e}{dC_L} = - \frac{\frac{dC_m}{dC_L}}{C_{m_{\delta_e}}} \quad (3.62)$$

Substituting equation 3.62 into 3.61 and solving for  $dC_m/dC_L$  corresponding to  $C_{L \text{ Max}}$

$$\frac{dC_m}{dC_{L \text{ Max}}} = \frac{(\delta_{e \text{ Zero Lift}} - \delta_{e \text{ Limit}})}{C_{L \text{ Max}}} C_{m_{\delta_e}} \quad (3.63)$$

Given a maximum  $C_L$  required for landing approach, equation 3.63 represents the maximum stability possible, or defines the most forward cg movement. A cg forward of this point prevents obtaining maximum  $C_L$  with limit elevator.

If a pilot were to maintain the  $C_{L_{Max}}$  for the approach, the value of  $dC_m/dC_L$  corresponding to this  $C_{L_{Max}}$  would be satisfactory. However, as is the case, the pilot desires additional  $C_L$  to maneuver as in flaring the aircraft. Additional elevator is required. This requirement then dictates a  $dC_m/dC_L$  less than the value required for  $C_{L_{Max}}$  only.

In addition to maneuvering the aircraft in the landing flare, the pilot must adjust for ground effect. The ground imposes a boundary condition which affects the downwash associated with the lifting action of the wing. This ground interference places the horizontal stabilizer at a reduced angle of attack. The equilibrium condition at the desired  $C_L$  is disturbed. To maintain the desired  $C_L$ , the pilot must increase  $\delta_e$  to obtain the original tail angle of attack. The maximum stability  $dC_m/dC_L$  must be further reduced to obtain additional  $\delta_e$  to counteract the reduction in downwash.

The three conditions that limit the amount of static longitudinal stability or most forward cg position are:

- The ability to land at high  $C_L$  in ground effect.
- The ability to maneuver at landing  $C_L$  (flare capability).
- The total elevator deflection available.

Figure 3.17A illustrates the limitations in  $dC_m/dC_{L_{Max}}$ .

### ● 3.10 STICK-FREE STABILITY

The name stick-free stability comes from the era of reversible control systems and is that variation related to the longitudinal stability which an aircraft would

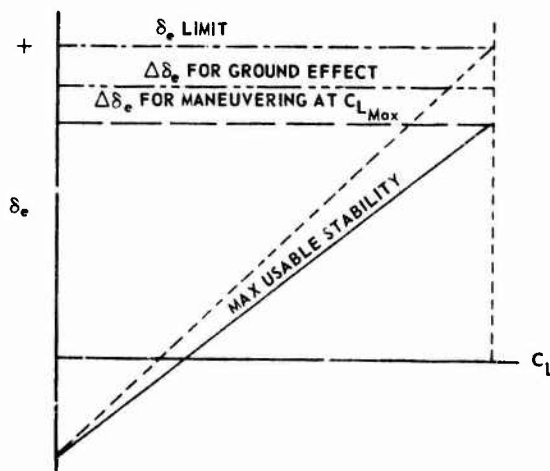
possess if the longitudinal control surface were left free to float in the slip stream. The control force variation with a change in airspeed is a measure of this stability.

If an airplane had an elevator that would float in the slip stream when the controls were free, then the change in the dynamic pressure pattern of the stabilizer would cause a change in the stability level of the airplane. The change in the stability contribution of the tail would be manifested by the floating characteristics of the elevator. Thus, the stick-free stability would depend upon the elevator hinge moments, control friction, or any device that would affect the moment of the elevator.

An airplane with an irreversible control system has very little tendency for its elevator to float. Yet the control forces presented to the pilot during flight, even though artificially produced, appear to be the effects of having a free elevator. If the control feel system can be altered artificially, then the pilot will see only good handling qualities and be able to fly

FIGURE 3.17a

LIMITATIONS ON  $dC_m/dC_{L_{Max}}$



what would normally be an unsatisfactory flying machine.

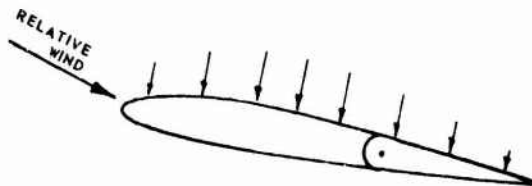
Stick-free stability can be analyzed by considering the effect of freeing the elevator of a tail-to-the-rear aircraft with a reversible control system. In this case the stick free stability would be indicated by the stick forces required to maintain the airplane in equilibrium at some speed other than trim.

The change in stability due to freeing the elevator, is a function of the floating characteristics of the elevator. The floating characteristics depend upon the elevator hinge moments which depend upon the change in pressure distribution over the elevator associated with changes in elevator deflection and tail angle of attack.

The analysis will look at the effect that pressure distribution has on the elevator hinge moments, the floating characteristics of the elevator, and then the effects of freeing the elevator.

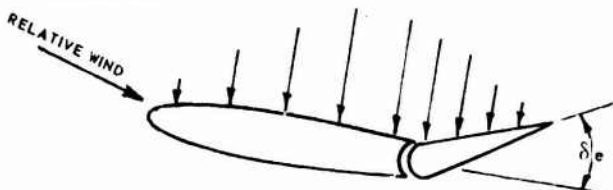
For a standard stable tail to the rear airplane, the pressure distribution would produce a downward load on the tail.

FIGURE 3.18



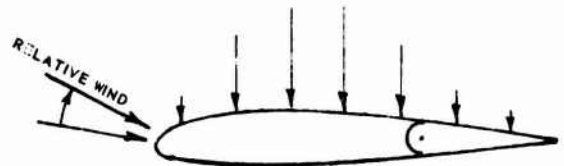
When the elevator is deflected the pressure distribution is changed.

FIGURE 3.19



When the stabilizer angle of attack is changed the pressure distribution is also changed.

FIGURE 3.20

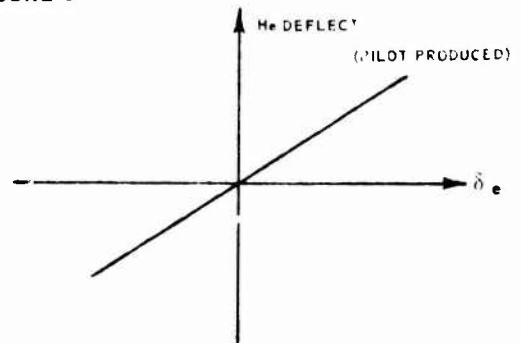


When the pressure distribution is changed, the hinge moments are changed. In order to deflect the elevator, the pilot had to apply a force to the stick and create a moment on the elevator hinge. The elevator hinge moment the pilot applied is now balanced by a moment caused by the pressure distribution on the control surface, and the elevator remains in the deflected position.

The pilot normally pulls back on the stick in order to produce a pitchup moment on the airplane. The hinge moment produced tends to move the control such that a positive moment on the airplane results. Therefore, the hinge moment is called positive. The pilot applies a positive moment to move the elevator. The pressure distribution produces a negative moment that opposes that of the pilot.

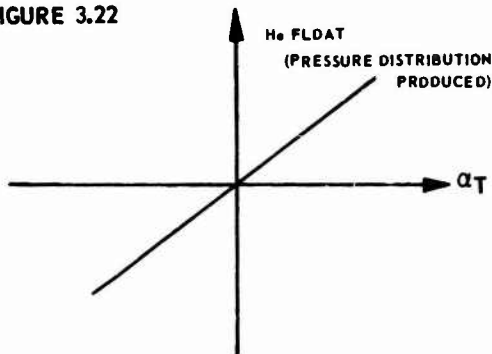
A plot of the pilot's hinge moment to deflect the elevator would be:

FIGURE 3.21



The hinge moment produced by the pressure distribution would be as shown in figure 3.22.

FIGURE 3.22

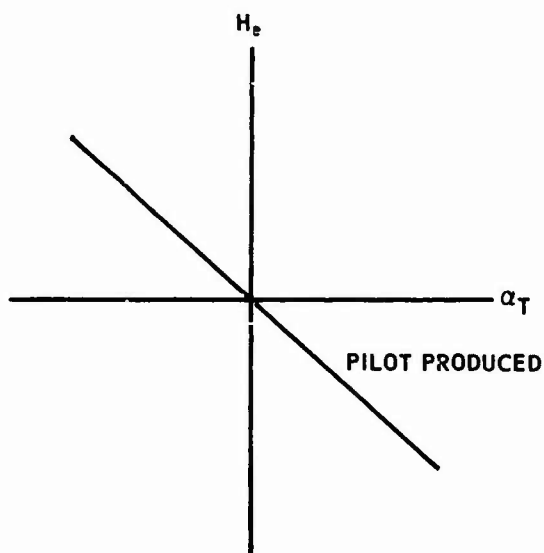


When the stabilizer angle of attack ( $\alpha_T$ ) is changed, the pilot must produce a control force in order to keep the elevator from floating in the slip stream.

Normally as the angle of attack is increased the elevator would tend to float up and the pilot would have to apply a negative push force in order to keep the stick from moving.

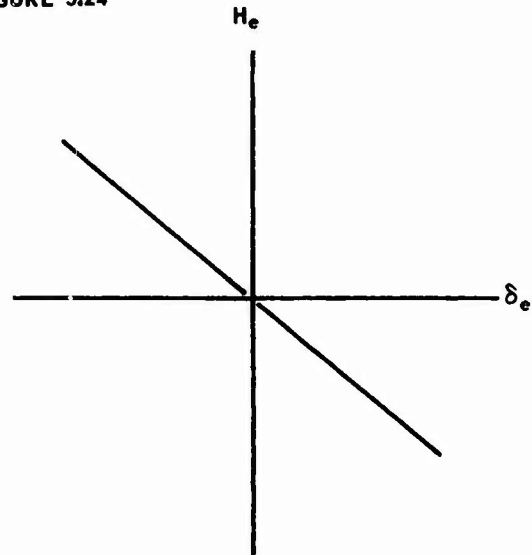
The hinge moment produced by the pilot to maintain trim deflection would be:

FIGURE 3.23



The hinge moment produced by the pressure distribution to float the elevator would be as shown in figure 3.24.

FIGURE 3.24



If we consider the moments produced by the pressure distribution on the elevator only, then we could analyze the floating characteristics of the elevator.

The hinge moments can be put in coefficient form in much the same manner as the airplane's aerodynamic moments. The  $H_e$  Restore Slope due to elevator deflection in coefficient form would be:

$$\frac{\partial C_h}{\partial \delta_e} \text{ Restore} = C_{h\delta} \quad (3.64)$$

The  $H_{e\text{Float}}$  slope due to angle of attack change in coefficient form would be:

$$\frac{\partial C_h}{\partial \alpha_e} \text{ Float} = C_{h\alpha} \quad (3.65)$$

Examining a floating elevator, it is seen that the total hinge moment is a function of elevator deflection, angle of attack, and mass distribution.

$$H_e = f(\delta_e, \alpha_T, W) \quad (3.66)$$

If the elevator is held at zero elevator deflection and zero angle of attack there may be some residual aerodynamic hinge moment  $Ch_0$ . The total hinge moment where  $W$  = weight of the elevator would be:

$$C_h = C_{h_0} + C_{h_\alpha} \alpha_T + C_{h_\delta} \delta_e + \frac{W}{qS} \frac{X}{c} \quad (3.67)$$

The weight effect is usually eliminated by mass balancing the elevator. Proper design of a symmetrical airfoil will cause  $Ch_0$  to be negligible.

When the elevator assumes its equilibrium position the total hinge moments will be zero and solving for the elevator deflection at this floating position.

$$\delta_{e\text{Float}} = - \frac{C_{h_\alpha}}{C_{h_\delta}} \alpha_T \quad (3.68)$$

The stability of the aircraft with the elevator free is going to be affected by this floating position.

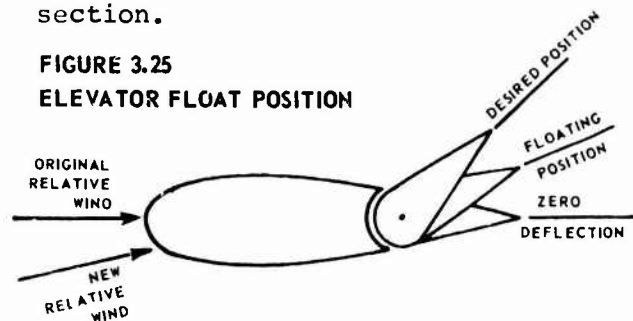
If the pilot desires to hold a new angle of attack from trim, he will have to deflect the elevator from this floating position to the position desired.

The floating position will greatly affect the forces the pilot is required to use. If the ratio  $Ch_\alpha/Ch_\delta$  can be adjusted, then the forces the pilot is required to use can be controlled.

If  $Ch_\alpha/Ch_\delta$  is small, then the elevator will not float very far and the stick-free stability characteristics will be much the same as those with the stick-fixed. But  $Ch_\delta$  must be small or the stick forces

required to hold deflection will be unreasonable. The values of  $Ch_\alpha$  and  $Ch_\delta$  can be controlled by aerodynamic balance. Types of aerodynamic balancing will be covered in a later section.

FIGURE 3.25  
ELEVATOR FLOAT POSITION



### 3.11 STICK-FREE STABILITY EQUATIONS

Stick free stability may be considered the summation of the stick-fixed stability and the contribution to stability of freeing the elevator.

$$\frac{dC_m}{dC_{L\text{Stick-Free}}} = \frac{dC_m}{dC_{L\text{Stick-Fixed}}} + \frac{dC_m}{dC_{L\text{Freeing Elevator}}} \quad (3.69)$$

Solving first for the effect to stability of freeing the elevator,

$$\frac{dC_m}{dC_{L\text{Free Elev.}}} = \frac{dC_m}{d\delta_e} \frac{d\delta_e}{dC_L} = C_{m_{\delta_e}} \frac{d\delta_e}{dC_L} \quad (3.70)$$

The stability contribution of the free elevator depends upon the elevator floating position. Equation 3.68 relates this position.

$$\delta_{e\text{Float}} = - \frac{C_{h_\alpha}}{C_{h_\delta}} \alpha_T \quad (3.71)$$

Substituting for  $\alpha_T$  from figure 3.10,

$$\delta_{e_{\text{Float}}} = - \frac{C_{h_\alpha}}{C_{h_\delta}} (\alpha_w - i_w + i_T - \epsilon) \quad (3.72)$$

Taking the derivative of equation 3.72 with respect to  $C_L$ ,

$$\frac{d\delta_e}{dC_L} = - \frac{C_{h_\alpha}}{C_{h_\delta}} \frac{\left(1 - \frac{d\epsilon}{d\alpha}\right)}{a_w} \quad (3.73)$$

Solving for  $dC_m/dC_{L_{\text{Free Elev.}}}$  in equation 3.70 and substituting the expression for elevator power,

$$C_{m_{\delta_e}} = - a_T \tau V_H \eta_T \quad (3.74)$$

$$\frac{dC_m}{dC_{L_{\text{Free Elevator}}}} = - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}}\right) \quad (3.75)$$

Substituting equation 3.75 and equation 3.33a ( $dC_m/dC_{L_{\text{Fixed}}}$ ) into equation 3.69, the stick-free stability becomes

$$\frac{dC_m}{dC_{L_{\text{Stick Free}}}} = \frac{x_w}{c} + \frac{dC_m}{dC_{L_{\text{Fus}}}} - \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}}\right) \quad (3.76)$$

The difference between stick-fixed and stick-free stability is the multiplier in equation 3.76,  $(1 - \tau C_{h_\alpha}/C_{h_\delta})$ , called the "free elevator factor" and which is designated  $F$ . The magnitude and sign of

$F$  depends on the relative magnitudes of  $\tau$  and the ratio of  $C_{h_\alpha}/C_{h_\delta}$ . An elevator with only slight floating tendency has a small  $C_{h_\alpha}/C_{h_\delta}$  giving a value of  $F$  around unity. The stick fixed and stick free stability are practically the same. If the elevator has a large floating tendency (ratio of  $C_{h_\alpha}/C_{h_\delta}$  large), the stability contribution of the horizontal tail is reduced materially ( $dC_m/dC_{L_{\text{Free}}}$  is less negative).

For instance, a ratio of  $C_{h_\alpha}/C_{h_\delta} = -2$  and a  $\tau$  of  $-0.5$ , the floating elevator can obviate the whole tail contribution to stability. Generally, freeing the elevator causes a destabilizing effect. With elevator free to float, the aircraft is less stable.

The stick-free neutral point,  $h'_n$ , is that cg position at which  $dC_m/dC_{L_{\text{Free}}}$  is zero. Continuing as in the stick-fixed case, the stick-free neutral point is,

$$h'_n = \frac{x_{ac}}{c} - \frac{dC_m}{dC_{L_{\text{Fus}}}} + \frac{a_T}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) F \quad (3.77)$$

and

$$\frac{dC_m}{dC_{L_{\text{Free}}}} = h - h'_n \quad (3.78)$$

The stick-free static margin is defined as,

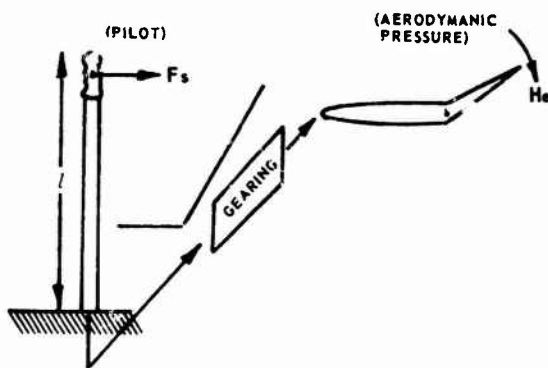
$$\text{Static Margin} = h'_n - h \quad (3.79)$$

### ● 3.12 STICK-FREE FLIGHT TEST RELATIONSHIP

As was done for stick-fixed stability, a flight test relationship is required that will relate measurable flight test parameters with the stick-free stability of

the aircraft  $dC_m/dC_{L_{Free}}$ . This relationship may be developed with reference to figure 3.26.

FIGURE 3.26  
ELEVATOR-STICK GEARING



The pilot holds a stick deflected with a stick force  $F_s$ . The control system transmits the moment from the pilot through the gearing to the elevator. The elevator deflects and the aerodynamic pressure produces a hinge moment at the elevator that exactly balances the moment produced by the pilot with force  $F_s$ .

$$F_s \ell = - G' H_e \quad (3.80)$$

If the length  $\ell$  is included with the gearing, the stick force becomes,

$$F_s = - G' H_e \quad (3.81)$$

The hinge moment  $H_e$  may be written,

$$H_e = C_{h_e} q S_e c_e \quad (3.82)$$

Equation 3.81 then becomes,

$$F_s = - G C_{h_e} q S_e c_e \quad (3.83)$$

Substituting

$$C_{h_e} = C_{h_o} + C_{h_{\alpha_T}} \alpha_T + C_{h_{\delta_e}} \delta_e + C_{h_{\delta_T}} \delta_T \quad (3.84)$$

where  $C_{h_{\delta_T}} \delta_T$  represents the tab contribution for an elevator with tab

where

$$\delta_e = \delta_{e_{zero lift}} + \frac{d\delta_e}{dC_L} C_L \quad (3.95)$$

$$\alpha_T = \alpha_w - i_w + i_T - \epsilon \quad (3.86)$$

Equation 3.83 may be written,

$$F_s = \underbrace{- G S_e c_e}_{A} q \left( \underbrace{C_{h_o} + C_{h_{\alpha}} (\alpha_{OL} - i_w + i_T) + C_{h_{\delta_e}} \delta_e}_{B} + C_{h_{\delta_T}} \delta_T - \frac{C_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \right) \quad (3.87)$$

Rewriting equation 3.87 with the above substitutions,

$$F_s = A q \left( B + C_{h_{\delta_T}} \delta_T - \frac{C_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \right) \quad (3.88)$$

Writing equation 3.88 as a function of airspeed and substituting for unaccelerated flight,  $C_L q = W/S$  and using equivalent airspeed,  $V_e$ ,

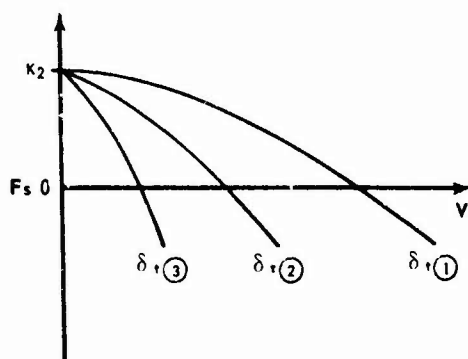
$$F_s = \frac{1}{2} \rho_o V_e^2 A \left( B + C_{h_{\delta_T}} \delta_T - \frac{C_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \right) \quad (3.89)$$

Simplifying equation 3.89 by combining constant terms,

$$F_s = K_1 V^2 + K_2 \quad (3.90)$$

$K_1$  contains  $\delta_T$  which determines trim speed.  $K_2$  contains  $dC_m/dC_{L_{Free}}$ . Equation 3.90 gives a relationship between an inflight measurement of stick force gradient and stick free stability. The equation is plotted in figure 3.27.

FIGURE 3.27  
STICK FORCE VERSUS AIRSPEED



The plot is made up of a constant force springing from the stability term plus a variable force proportional to the velocity squared, introduced through some constants and the tab term  $C_{h\delta_T}\delta_T$ . Equation 3.90 introduces an interesting fact that the stick force variation with airspeed is apparently dependent on the first term only and independent in general of the stability level. That is, the slope of the curve  $F_s$  versus  $V$  is not a direct function of  $dC_m/dC_{L_{Free}}$ . If the derivative of equation 3.89 is taken with respect to  $V$ , the second term containing the stability drops out.

$$\frac{dF_s}{dV} = \rho_o V A (B + C_{h\delta_T} \delta_T) \quad (3.91)$$

However,  $dF_s/dV$  may be made a function of the stability term

using another approach. The tab setting  $\delta_T$  in equation 3.89 should be adjusted to obtain  $F_s = 0$ . This is  $\delta_T$  for trim velocity, i.e.,

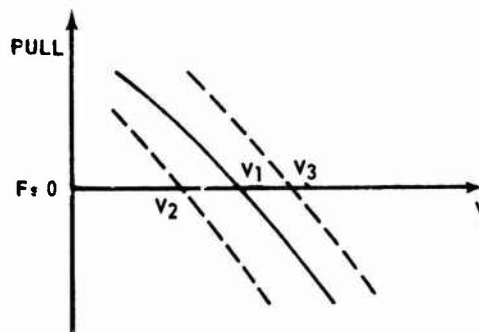
$$\delta_{T_{F_s=0}} = f(V_{Trim}, \frac{dC_m}{dC_{L_{Free}}}) \quad (3.92)$$

This value of  $\delta_{T_{F_s=0}}$  is then substituted into equation 3.91 so that,

$$\frac{dF_s}{dV_{Trim}} = f(V_{Trim}, \frac{dC_m}{dC_{L_{Free}}}) \quad (3.93)$$

Thus it appears that if an aircraft is flown at three cg locations and  $dF_s/dV_{Trim}$  through the same trim speed each time is determined, then one could extrapolate or interpolate to determine the stick-free neutral point  $h_n$ . Unfortunately, if there is a significant amount of friction in the control system, it is impossible to precisely determine this trim speed. In order to investigate briefly the effects of friction on the longitudinal control system, suppose that the aircraft represented in figure 3.28 is perfectly trimmed at  $V_1$  (i.e.,  $\delta_e = \delta_{e1}$  and  $\delta_T = \delta_{T1}$ ). If the airspeed is decreased or increased with no change to the trim setting, the friction in the control system will

FIGURE 3.28  
CONTROL SYSTEM FRICTION



prevent the elevator from returning all the way back to  $\delta_{e1}$  when the controls are released. The aircraft will return only to  $V_2$  or  $V_3$ . With the trim tab at  $\delta_{T1}$ , the aircraft is content to fly at any speed between  $V_2$  and  $V_3$ . The more friction that exists in the system, the wider this speed range becomes.

Therefore, if there is a significant amount of friction in the control system, it becomes impossible to say that there is one exact speed for which the aircraft is trimmed. Equation 3.93 then, is something less than perfect for predicting the stick-free neutral point of an aircraft. To reduce the undesirable effect of friction in the control system, a different approach is made to equation 3.88.

If equation 3.88 is divided by the dynamic pressure  $q$ , then,

$$\frac{F_s}{q} = A(B + C_{h_{\delta_T}} \delta_T) - \frac{AC_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \quad (3.94)$$

Differentiating with respect to  $C_L$ ,

$$\frac{dF_s/q}{dC_L} = - \frac{AC_L C_{h_{\delta_e}}}{C_{m_{\delta_e}}} \frac{dC_m}{dC_{L_{Free}}} \quad (3.95)$$

or

$$\frac{dF_s/q}{dC_L} = f \left( \frac{dC_m}{dC_{L_{Free}}} \right) \quad (3.96)$$

Trim velocity is now eliminated from consideration, and the prediction of stick-free neutral point  $h'_n$  is more exact. A plot of  $dF_s/q/dC_L$  versus cg position may be extrapolated to obtain  $h'_n$ .

### 3.13 APPARENT STICK-FREE STABILITY

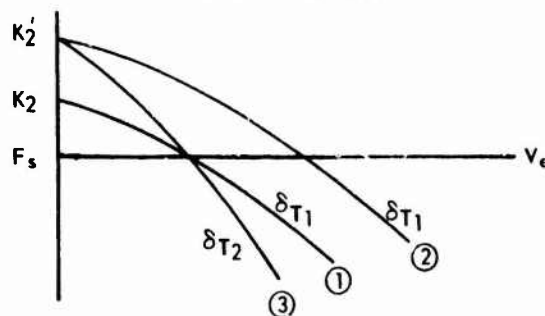
Speed stability or stick force gradient  $dF_s/dV$  in most cases does not reflect the actual stick-free stability  $dC_m/dC_{L_{Free}}$  of an aircraft. In fact this apparent stability  $dF_s/dV$  may be quite different from the actual stability of the aircraft. Where the actual stability of the aircraft may be marginal ( $dC_m/dC_{L_{Free}}$  small), or even unstable ( $dC_m/dC_{L_{Free}}$  positive), the apparent stability  $dF_s/dV$  may be such as to make the aircraft quite acceptable. In flight, the test pilot feels and evaluates the apparent stability of the aircraft and not the actual stability  $dC_m/dC_{L_{Free}}$ .

The apparent stability  $dF_s/dV$  is affected by:

1. Changes in  $dC_m/dC_{L_{Free}}$
2. Aerodynamic balancing
3. Downsprings and/or bob weights.

The apparent stability or the stick force gradient through a given trim speed increases if  $dC_m/dC_{L_{Free}}$  is made more negative. The constant  $K_2$  of equation 3.90 is made more positive and in order for the stick force curve to continue to pass through the desired trim speed, a more positive tab selection is required. An aircraft operating at a certain cg with a tab setting  $\delta_{T1}$  is shown on figure 3.29, line 1.

FIGURE 3.29  
EFFECT ON APPARENT STABILITY



If  $dC_m/dC_{L_{Free}}$  is increased by moving the cg forward,  $K_2$ , which is a function of  $dC_m/dC_{L_{Free}}$  in equation 3.89 becomes more positive or increases. The new equation becomes,

$$F_s = K_1 V_e^2 + K_2' \quad (3.97)$$

This equation plots as line 2 in figure 5.7. The aircraft with no change in tab setting  $\delta T_1$  operates on line 2 and is trimmed to  $V_2$ . Stick forces at all airspeeds have increased. At this juncture, although the actual stability  $dC_m/dC_{L_{Free}}$  has increased, there has been no effect on the stick force gradient or apparent stability. (The slopes of line 1 and line 2 being the same.) So as to retrim to the original trim airspeed  $V_1$ , the pilot applies additional nose up tab to  $\delta T_2$ . The aircraft is now operating in line 3. The stick force gradient through  $V_1$  has increased because of an increase in the  $K_1$  term in equation 3.89. The apparent stability  $dF_s/dV$  has increased.

The same effect on apparent stability as cg movement may be obtained by means of aerodynamic balancing. This is a design means of controlling the hinge moment coefficients,  $Ch_\alpha$  and  $Ch_\delta$ . The primary reason for aerodynamic balancing is to increase or reduce the hinge moments and, in turn, the control stick forces. Changing  $Ch_\delta$ , affects the stick forces as seen in equation 3.89. In addition to the influence on hinge moments, aerodynamic balancing may very well affect the real and apparent stability of the aircraft. Assuming that the restoring hinge moment coefficient  $Ch_\delta$  is increased by an appropriate aerodynamic balanced control surface, the ratio of  $Ch_\alpha/Ch_\delta$  in stability equation 3.98 is decreased.

$$\frac{dC_m}{dC_{L_{Free}}} = \frac{x_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H n_T \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(1 - \tau \frac{C_{h\alpha}}{C_{h\delta}}\right) \quad (3.98)$$

The combined increase in  $dC_m/dC_{L_{Free}}$  and  $Ch_\delta$ , increases the  $K_2$  term in equation 3.90 since

$$K_2 = -A \frac{W}{S} \frac{C_{h\delta}}{C_{m\delta_e}} \frac{dC_m}{dC_{L_{Free}}} \quad (3.99)$$

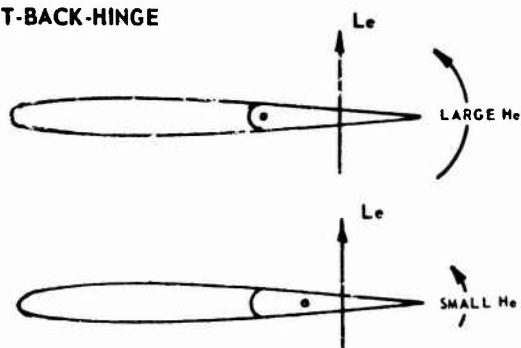
Figure 3.29 shows the effect of increased  $K_2$ . The apparent stability is not affected by the increase in  $K_2$  while the aircraft retrimms at  $V_2$ . However, once the aircraft is retrimmed to the original airspeed  $V_1$  by increasing the tab setting to  $\delta T_2$ , the apparent stability is increased.

Types of aerodynamic balancing used to control the hinge moment coefficients are as follows:

#### Set-Back-Hinge:

Perhaps the simplest method of reducing the aerodynamic hinge moments is simply to move the hinge line rearward. Thus the hinge moment is reduced because of the moment arm between the elevator lift and the elevator hinge line is reduced. (One may arrive at the same conclusion by arguing that part of the elevator lift acting behind the hinge line has been reduced, while that in front of the hinge line has been increased.) The net result is a reduction in the absolute value of both  $Ch_\alpha$  and  $Ch_\delta$ . In fact if the hinge line is set back far enough, the sign of both derivatives can be changed.

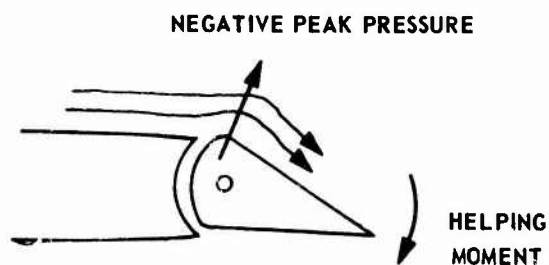
FIGURE 3.30  
SET-BACK-HINGE



Overhang Balance:

This method is simply a special case of set-back hinge in which the elevator is designed so that when the leading edge protrudes into the airstream, the local velocity is increased significantly; causing an increase in negative pressure at that point. This negative pressure peak creates a hinge moment which opposes the normal restoring hinge moment, reducing  $Ch_\delta$ . Figure 3.31 shows an elevator with an overhang balance.

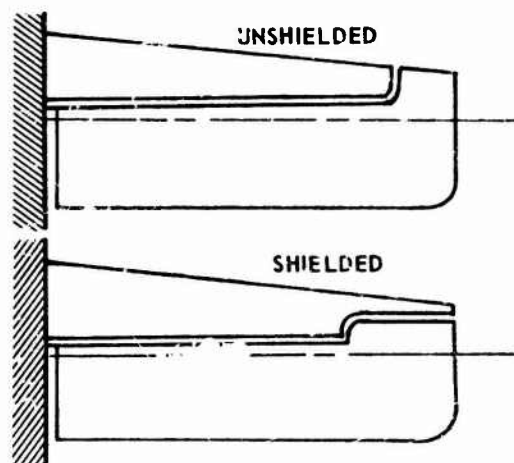
FIGURE 3.31  
OVERHANG BALANCE



Horn Balance:

The horn balance works on the same principle as the set-back hinge, i.e., to reduce hinge moments by increasing the area forward of the hinge line. The horn balance, especially the unshielded horn, is very effective in reducing  $Ch_\alpha$  and  $Ch_\delta$ . This arrangement shown in figure 3.32, is also a handy way of improving the mass balance of the control surface.

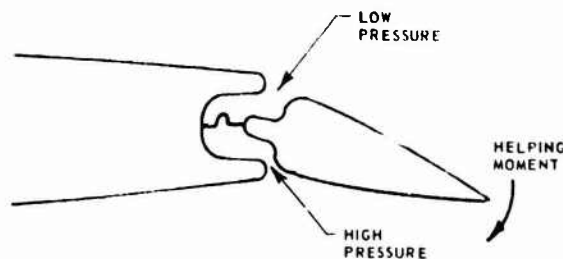
FIGURE 3.32  
HORN BALANCE



Internal Balance or Internal Seal:

The internal seal allows the negative pressure on the upper surface and the positive pressure on the lower surface to act on an internal sealed surface forward of the hinge line in such a way that a helping moment is created, again opposing the normal hinge moments. As a result, the absolute values of  $Ch_\alpha$  and  $Ch_\delta$  are both reduced. This method is good at high indicated airspeeds but is sometimes troublesome at high Mach numbers.

FIGURE 3.33  
INTERNAL SEAL



### Elevator Modifications:

#### Bevel Angle on Top or on Bottom of the Stabilizer.

This device, which causes flow separation on one side but not on the other, reduced the absolute values of  $Ch_\alpha$  and  $Ch_\delta$ .

#### Trailing Edge Strips.

This device, found on the B-57, increases both  $Ch_\alpha$  and  $Ch_\delta$ . In combination with a balance tab, trailing edge strips produce a very high positive  $Ch_\alpha$ , but still a low  $Ch_\delta$ . This results in what is called a favorable "Response Effect," i.e., it takes a lower control force to hold a deflection than was originally required to produce it.

#### Convex Trailing Edge.

This type surface produces a more negative  $Ch_\delta$ , but tends as well to produce a dangerous short-period oscillation.

#### Tabs:

A tab is simply a small flap which has been placed on the trailing edge of a larger one. The tab greatly modifies the flap hinge moments but has only a small effect on the lift of the elevator or the entire airfoil. Tabs in general are designed to modify stick-forces and therefore  $Ch_\delta$  but will not affect  $Ch_\alpha$ . A positive tab deflection is one which will tend to move the elevator in a positive direction.

#### Trim Tab.

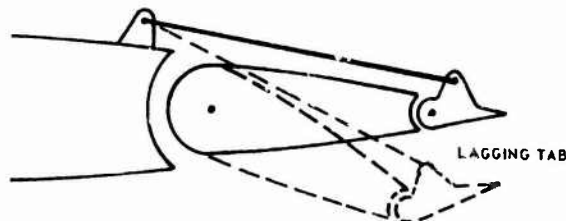
A tab which is controlled by a switch or control separate from the normal cockpit pilot control is called a trim tab. The purpose of the trim tab is to reduce the elevator hinge moment and, therefore, the stick force to zero for a given flight condition. A satis-

factory trim tab should be able to accomplish this throughout the aircraft flight envelope. Ordinarily, a trim tab will not significantly vary  $Ch_\alpha$  or  $Ch_\delta$ . The functions of the spring and trim or balance and trim tabs may be combined in a single tab. Another method of trimming an aircraft is the use of an adjustable horizontal stabilizer. Normally the trim tab or horizontal stabilizer setting will have a small effect on stability.

#### Balance Tab.

A balance tab is a simple tab which is mechanically geared to the elevator so that a certain elevator deflection produces a given tab deflection. If the tab is geared to move in the same direction as the surface, it is called a leading tab. If it moves in the opposite direction, it is called a lagging tab. The purpose of the balance tab is usually to reduce the hinge moments and stick forces (lagging tab) at the price of a certain loss in control effectiveness. Sometimes, however, a leading tab is used to increase control effectiveness at the price of increased stick forces. The leading tab may also be used for the express purpose of increasing control forces. Thus  $Ch_\delta$  may be increased or decreased, while  $Ch_\alpha$  remains unaffected. If the linkage shown in figure 3.34 is made so that the length may be varied by the pilot, then the tab may also serve as a trimming device.

FIGURE 3.34  
BALANCE TAB



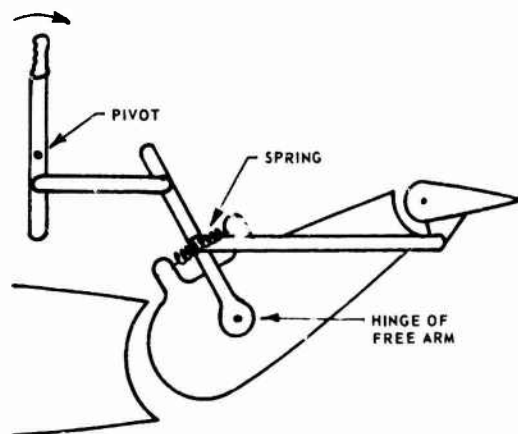
### Servo or Control Tab.

The servo tab is linked directly to the aircraft control system in such a manner that the pilot moves the tab and the tab moves the elevator, which is free to float. The summation of elevator hinge moments, therefore, always equals zero since the elevator will float until the hinge moment due to elevator deflection just balances out the hinge moments due to  $\alpha_s$  and  $\delta_t$ . The stick forces are now a function of the tab hinge moment or  $Ch_{\delta T}$ . Again  $Ch_{\alpha}$  is not affected.

### Spring Tab.

A spring tab is a lagging balance tab which is geared in such a way that the pilot receives the most help from the tab at high speeds where he needs it the most, i.e., the gearing is a function of dynamic pressure. The basic principles of its operation are:

FIGURE 3.35  
SPRING TAB

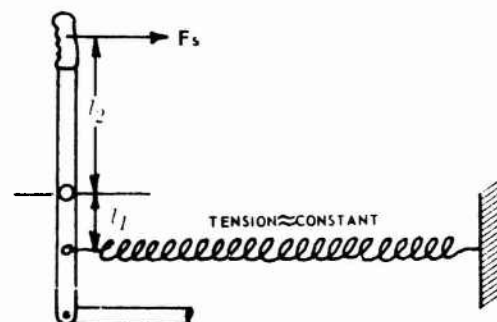


1. An increase in dynamic pressure causes an increase in hinge moment and stick force for a given control deflection.
2. The increased stick force causes an increased spring deflection and, therefore, an increased tab deflection.

3. The increased tab deflection causes a decrease in stick force. Thus an increased proportion of the hinge moment is taken up by the tab.
4. Therefore, the spring tab is a geared balance tab where the gearing is a function of dynamic pressure.
5. Thus the stick forces are more nearly constant over the speed range of the aircraft. That is, the stick force required to produce a given deflection at 300 knots is still greater than at 150 knots, but not by as much as before.
6. After full spring or tab deflection is reached the balancing feature is lost and the pilot must supply the full force necessary for further deflection. (This acts as a safety feature.)

Because of the very low force gradients in most modern aircraft at the aft center of gravity ( $dC_m/d\alpha_{CLFree}$  less negative), improvements in the stick-free longitudinal stability are obtained by devices which produce a constant pull force on the stick independent of airspeed which allows a more noseup tab setting and steeper stick force gradients. Two such gadgets for improving the stick force gradients are the downspring and bobweight. Both effectively increase the apparent stability of the aircraft.

FIGURE 3.36  
DOWNSPRING



### Downspring:

A virtually constant stick force may be demanded of the pilot by incorporating a downspring or bungee into the control system which tends to pull the top of the stick forward. From figure 3.36, the force required to counteract the spring is,

$$F_{s_{\text{Downspring}}} = T \frac{\ell_1}{\ell_2} = K_3 \quad (3.100)$$

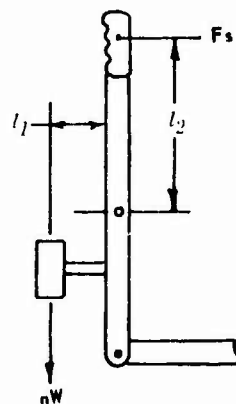
If the spring is a long one, the tension in it will be increased only slightly as the top moves rearward and can be considered to be constant.

The equation with the downspring in the control system becomes,

$$F_s = K_1 V_e^2 + K_2 + K_3_{\text{Downspring}} \quad (3.101)$$

As shown in figure 3.36, the apparent stability will increase when the aircraft is once again retrimmed to the original trim airspeed by increasing the tab setting. Note that the downspring increases apparent stability but does not affect the actual stability  $dC_m/dC_{L_{\text{Free}}}$  (no change to  $K_2$ ) of the aircraft.

FIGURE 3.37  
BOBWEIGHT



### Bobweight:

Another method of introducing a nearly constant stick force is by placing a bobweight somewhere in the control system which causes a constant moment which must be overcome by the pilot. The force which the pilot must apply to counteract the bobweight is,

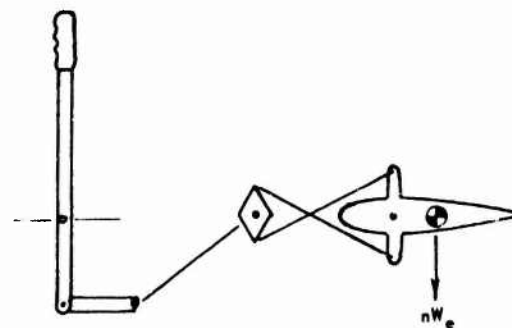
$$F_{s_{\text{Bobweight}}} = nW \frac{\ell_1}{\ell_2} = K_3 \quad (3.102)$$

Like the downspring the bobweight increases the stick force throughout the airspeed range and, at increased tab settings, the apparent stability or stick force gradient. The bobweight has no effect on the actual  $dC_m/dC_{L_{\text{Free}}}$  of the aircraft.

### Elevator Unbalance:

There are other devices which increase the stick force gradient through trim or apparent stability. The unbalance in the control system resulting from the center of gravity of the elevator falling aft of the hinge line is shown in figure 3.38.

FIGURE 3.38  
ELEVATOR UNBALANCE



From the figure it can be seen that an elevator cg behind the hinge line will tend to rotate the top of the stick forward. This must be counteracted by a positive pull stick force.

As the elevator is moved from the horizontal, the hinge moment is reduced by the cosine of the deflection angle; this moment remains virtually constant. Thus a forward hinge line which usually produces a destabilizing (positive)  $Ch_\alpha$  will also produce a "stabilizing" elevator unbalance.

Comment:

Since  $h_n^1$  is usually found by equation 3.96, it would be worthwhile to examine the effect of the stick force gradient  $dF_s/dV$  on this equation. Rewriting equation 3.88, with a downspring used as the control system gadget,

$$F_s = Aq(B + C_{h\delta_T}) - AC_L q \frac{C_{h\delta}}{C_{m\delta_e}} \frac{dC_m}{dC_{L_{Free}}} \quad \text{No Gadget} \\ + K_3' \quad \text{Gadget} \quad (3.103)$$

$$F_s/q = A(B + C_{h\delta_T}) - AC_L \frac{C_{h\delta}}{C_{m\delta_e}} \frac{dC_m}{dC_{L_{Free}}} \quad \text{No Gadget} \\ + \frac{K_3' C_L}{W/S} \quad (3.104)$$

$$\frac{dF_s/q}{dC_L} = K_2' \frac{dC_m}{dC_{L_{Free}}} + \frac{K_3'}{W/S} \quad \text{No Gadget} \quad (3.105)$$

Obviously the cg location at which  $dF_s/q/dC_L$  goes to zero will not be the true  $h_n^1$ . However, the only reason that the term  $dC_m/dC_{L_{Free}}$  was of interest in the first place was because it was proportional to the stick force gradient. The pilot is more interested in the apparent stability for the same reason. The fact that the addition to the stick-

free stability caused by this gadgetry is "artificial" rather than genuine is only of academic interest.

### 3.14 HIGH SPEED LONGITUDINAL STATIC STABILITY

The effects of high speeds (transonic and supersonic) on longitudinal static stability can be analyzed in the same manner as that done for subsonic speeds. The assumptions that were made for the incompressible flow are no longer valid and, therefore, cannot be neglected.

Compressibility associated with the transonic and supersonic speed regime has noticeable effect upon both the gust stability (longitudinal static stability  $Cm_{CL}$ ) and speed stability ( $F_s/V$ ). The gust stability depends mainly on the contributions to stability of the wing, fuselage, and tail in the stability equation below during transonic and supersonic flight.

$$\frac{dC_m}{dC_L} = \frac{X_w}{c} + \frac{dC_m}{dC_{L_{Fus}}} - \frac{a_T}{a_w} V_H \eta_T \left( 1 - \frac{ac}{d\alpha} \right) \quad (3.106)$$

The terms in the stability equation will be examined in turn.

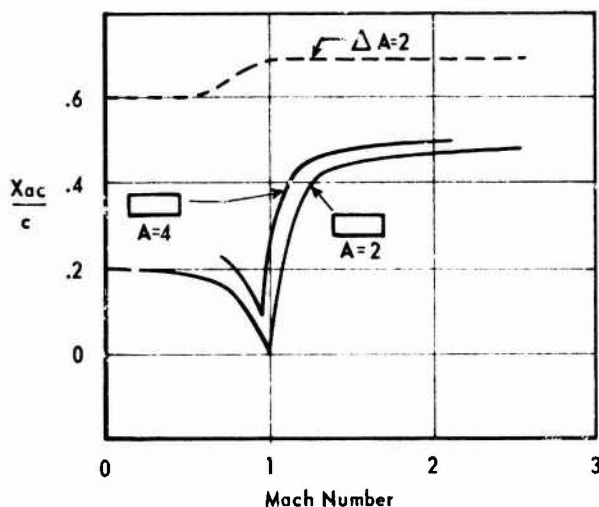
#### The Wing Contribution:

In subsonic flow or low Mach flight, the aerodynamic center is at the quarter chord. As subsonic Mach approaches unity or the transonic speed is approached, flow separation occurs behind the shock formations causing the aerodynamic center to move forward of the quarter chord position. The immediate effect is a reduction in stability since  $X_w/c$  increases. Following the flow separation behind the shocks at

positions of sonic speed, the flow pattern on the airfoil eventually transitions to supersonic flow. The shocks move off the surface and the wing recovers lift. The aerodynamic center now moves aft towards the 50-percent chord position. There is a sudden increase in the wing's contribution to stability since  $X_w/c$  is reduced (figure 3.1).

The extent of the aerodynamic center shift forward and rearward depends greatly on the aspect ratio of the aircraft. The shift is least for low aspect ratio aircraft. Among the plan forms, the rectangular wing has the largest shift for a given aspect ratio whereas the triangular wing has the least (figures 3.39 and 3.14).

FIGURE 3.39  
A.C. VARIATION WITH MACH



#### The Fuselage Contribution:

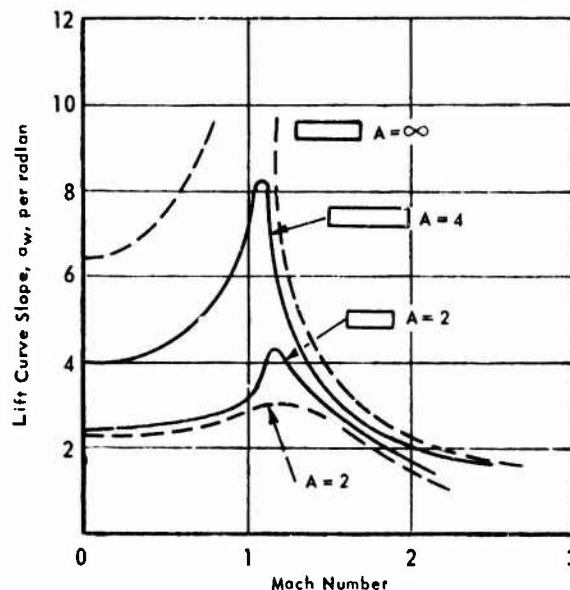
In supersonic flow the fuselage center of pressure moves forward causing a positive increase in the fuselage  $dC_m/dC_L$  or a destabilizing influence on the stability equation. The fuselage term variation with Mach number will be ignored.

#### The Tail Contribution:

The tail contribution to stability depends on the variation of lift curve slopes,  $a_w$  and  $a_T$ , plus downwash  $\epsilon$  with Mach during transonic and supersonic flight. It is expressed as:  $-a_T/a_w V_{HNT} (1 - d\epsilon/d\alpha)$

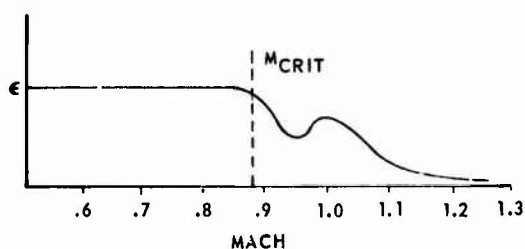
During subsonic flight  $a_T/a_w$  remains approximately constant. The slope of the lift curve,  $a_w$  varies as shown in figure 3.40. This variation of  $a_w$  in the transonic speed range is a function of geometry (i.e., aspect ratio, thickness, camber, and sweep). Limiting further discussion to aircraft designed for transonic flight or aircraft which employ airfoil shapes with small thickness to chord ratios, then  $a_w$  increases slightly in the transonic regime. For all airfoil shapes the value of  $a_w$  decreases as the airspeed increases supersonically. The  $a_T/a_w$  contribution is therefore destabilizing in the transonic regime and stabilizing in the supersonic regime.

FIGURE 3.40  
LIFT CURVE SLOPE VARIATION WITH MACH



The tail contribution is further affected by the variation in downwash,  $\epsilon$ , with Mach increase. The downwash at the tail is a result of the vortex system associated with the lifting wing. It is recognized that the tail location will have considerable influence as to the degree of variation of  $\delta_e$  with  $\Delta\epsilon$ . An aircraft such as the F-100 has a great deal more variation of  $\delta_e$  due to downwash effects than the F-104. Since downwash is a direct function of wing lift, a sudden loss of downwash occurs transonically with a resulting increase in tail angle of attack. The effect is to require the pilot to apply additional up elevator with increasing airspeed to maintain altitude. This additional up elevator contributes to speed instability. (Speed stability will be covered more thoroughly later.) Downwash variation with Mach is seen in figure 3.41.

FIGURE 3.41  
DOWNWASH VARIATION WITH MACH



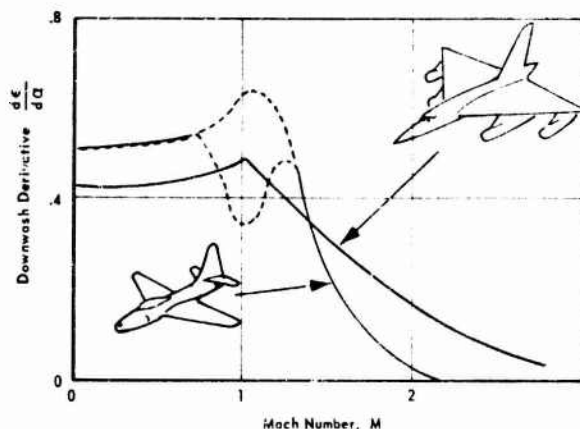
The variation of  $d\epsilon/d\alpha$  with Mach number greatly influences the aircraft's gust stability  $dC_m/dC_L$ . Recalling,

$$\epsilon = \frac{114.6 C_L}{\pi AR} \quad \text{where} \quad \frac{d\epsilon}{d\alpha} = \frac{114.6 a_w}{\pi AR} \quad (3.107)$$

Since the downwash angle behind the wing is directly proportional to the lift coefficient of the wing, it is apparent that the value of the derivative  $d\epsilon/d\alpha$  is a func-

tion of  $a_w$ . The general trend of  $d\epsilon/d\alpha$  is an initial increase with Mach starting at subsonic speeds. This increase follows a trend similar to but at a lesser slope than the increase of the lift curve slope,  $a_w$ , of the wing. Above Mach 1.0,  $d\epsilon/d\alpha$  decreases and approaches zero. This variation depends on the particular wing geometry of the aircraft. As shown in figure 3.42,  $d\epsilon/d\alpha$  may dip for thicker wing sections where considerable flow separation occurs. Again,  $d\epsilon/d\alpha$  is very much dependent on  $a_w$ .

FIGURE 3.42  
DOWNWASH DERIVATIVE vs MACH



For an aircraft designed for high speed flight, the variation of  $d\epsilon/d\alpha$  with increasing Mach number results in a slight destabilizing effect in the transonic regime and contributes to increased stability in the supersonic speed regime.

As the wing surface becomes a less efficient lifting surface, a loss of stabilator effectiveness is experienced in supersonic flight. The elevator power,  $C_m \dot{\delta}_e$ , increases as airspeed approaches Mach 1.0. Beyond Mach 1.0, elevator effectiveness decreases. Consequently, increase of elevator power causes a positive  $\Delta\delta_e$  contribution or again an indication of speed instability

as Mach 1.0 is approached. With decrease in elevator power, a negative  $\Delta\delta_e$  contribution once again produces speed stability. For the F-104 the relative order of magnitude of these values cause an initial increase in gust stability in the transonic regime followed by a steadily decreasing stability influence as  $C_{m\delta_e}$  approaches zero.

$$\frac{dC_m}{dC_{L_{Tail}}} = \frac{C_{m\delta_e}}{a_w} \left( 1 - \frac{d\epsilon}{d\alpha} \right) \quad (3.108)$$

The overall effect of transonic and supersonic flight on gust stability or  $dC_m/dC_L$  is shown in figure 3.43. Static longitudinal stability increases transonically and then decreases supersonically. The speed stability of the aircraft is affected as well. Recalling the pitching moment coefficient equation,

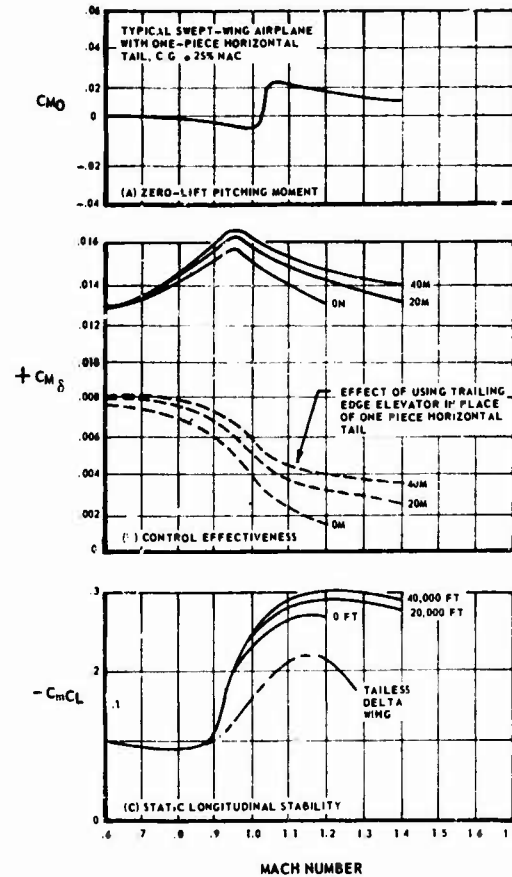
$$\begin{aligned} \Delta C_m = & C_{m_0} + C_{m_\alpha} \Delta\alpha + C_{m_\alpha^*} \Delta\alpha^* + C_{m_{\delta_e}} \Delta\delta_e \\ & + C_{m_v} \Delta V + C_{m_q} \Delta q \end{aligned} \quad (3.109)$$

and since  $C_{m_{C_L}} = \frac{1}{a} C_{m_\alpha}$ , then:  
Assuming no change in speed or pitch rate, and since under compressibility  $C_{m_0}$  is not zero, the elevator required to maintain steady flight is:

$$\Delta\delta_e = - \frac{C_{m_0}}{C_{m_{\delta_e}}} - \frac{C_{m_{C_L}}}{C_{m_{\delta_e}}} \Delta C_L \quad (3.110)$$

Speed stability depends on the variations of  $\delta_e$  with transonic and supersonic speeds and according to equation 3.110, depends on how  $C_{m_0}$ ,  $C_{m_{\delta_e}}$ , and  $C_{m_{C_L}}$  vary.

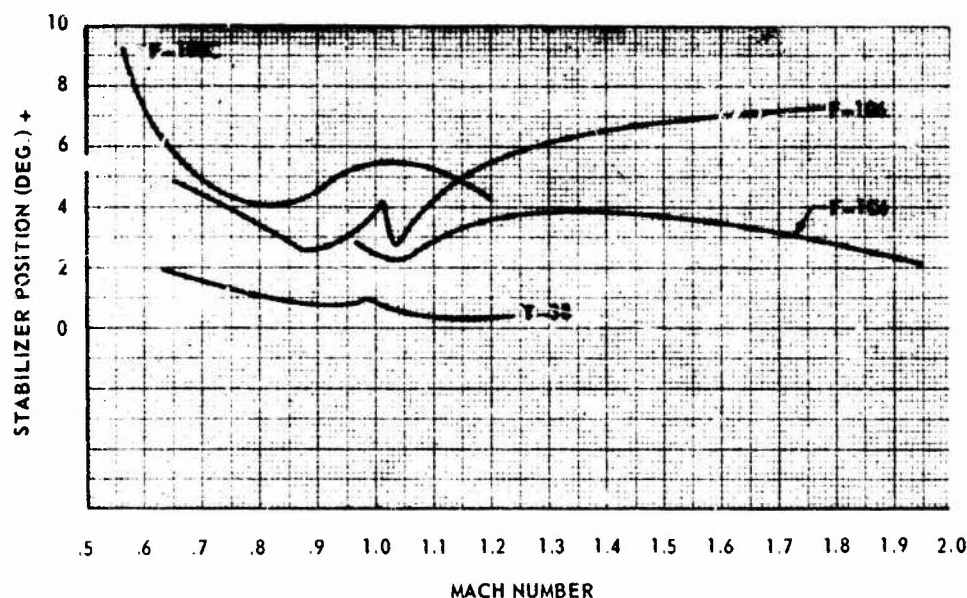
FIGURE 3.43  
MACH VARIATIONS ON  $C_{m_0}$ ,  $C_{m_{\delta_e}}$ , AND  $C_{m_{C_L}}$



If equation 3.110 is analyzed using the plots in figure 3.43, speed instability during transonic flight becomes obvious. The value of  $-C_{m_0}/C_{m_{\delta_e}}$  increases from approximately zero in the subsonic range to some positive value as the aircraft passes through Mach 1.0. The value of  $C_{m_{C_L}}/C_{m_{\delta_e}}$  increases to a very large number in comparison to  $C_{m_0}/C_{m_{\delta_e}}$  through this same range. The result is a positive  $\Delta\delta_e$  or a reversal of elevator deflection with increasing airspeed. This manifests itself as a relaxation of forward pressure or even a pull force to maintain attitude or prevent a nose down tendency. As the aircraft speed increases to supersonic speed,  $\Delta\delta_e$  again becomes nega-

tive and the pilot regains speed stability or decreasing  $\delta_e$  with increasing airspeed. The actual results of some aircraft flown in this range are shown in figure 3.44.

FIGURE 3.44  
 $\delta_e$  vs MACH



Whether the speed instability or reversal in elevator deflections and stick forces are objectionable, depends on many factors such as magnitude of variation, length of time required to transverse the region of instability, control system characteristics, and conditions of flight. It is impossible for the engineer to determine from data plots if the degree of instability is acceptable. The pilot is the only one capable of evaluating these effects.

In the F-100C, a pull force of 14 pounds was required when accelerating from Mach 0.87 to Mach 1.0. The test pilot described this trim change as disconcerting while attempting to maneuver the aircraft in this region and recommended that a "q" or Mach sensing device be installed to eliminate

this characteristic. Consequently, a mechanism was incorporated to automatically change the artificial feel gradient as the aircraft accelerates through the transonic range. Also, the longitudinal trim is automatically changed in this region by the use of a "Mach Trimmer."

In the F-104, the test pilot stated that transonic trim changes required an aft stick movement with increasing speed and a forward stick movement when decreasing speed, but described this speed instability as acceptable.

In the F-106 the pilot stated that the 1.0 to 1.1 Mach region is characterized by a moderate trim change necessitating pilot technique to avoid large variations in altitude during accelerations. Minor trim changes are encountered up to

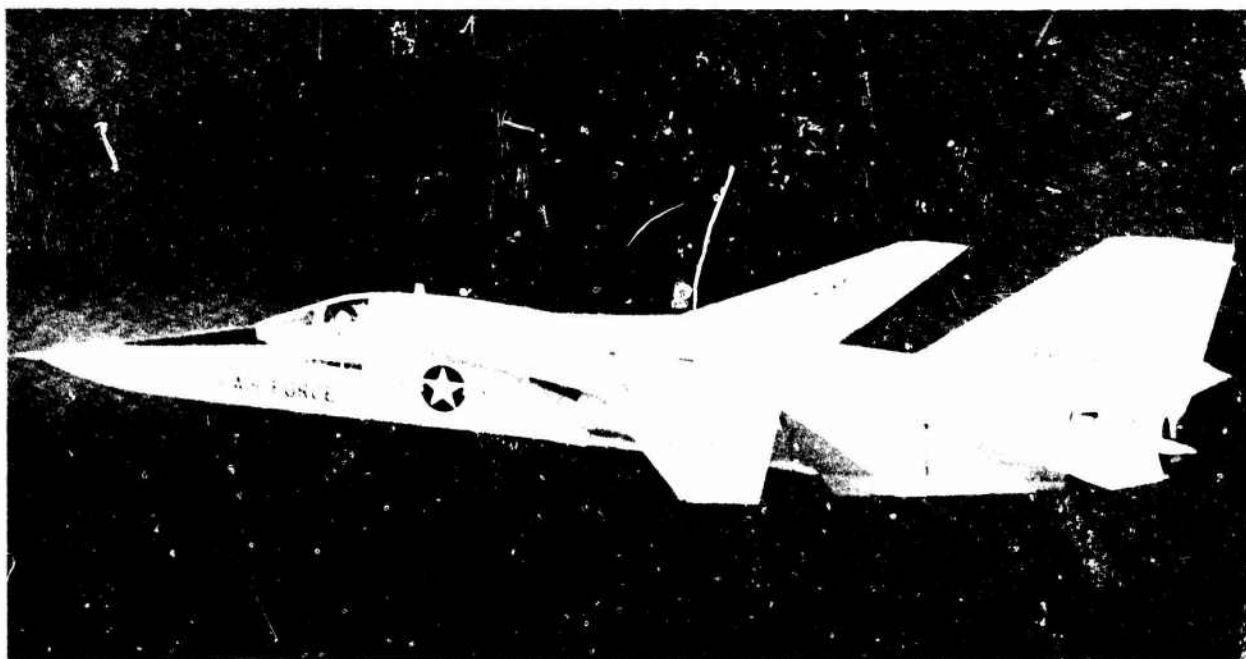
Mach 1.35. His report concluded that the speed instabilities were not unsatisfactory.

In the T-38 which embodies the latest design concept, a departure is noted from the low tail configuration difficulties where the pilot described the transonic trim change as being hardly perceptible.

Aircraft design considerations are, of course, influenced by the stability aspects of high speed flight. It is desirable to design an aircraft where trim changes through transonic speeds are small. A flat wing without camber, twist, or incidence or a low aspect ratio wing and tail provide values of  $X_w/c$ ,  $a_w$ ,  $a_T$ , and  $d\epsilon/d\alpha$  which vary

minimally over the Mach number range. An all-moving tail (slab) for control gives negligible variation of  $\tau$  with Mach and maximum control effectiveness. A full power, irreversible control system is desirable to counteract the erratic changes in pressure distribution which affect  $Ch_\alpha$  and  $Ch_{\delta_e}$ .

In the transonic speed regime the meaning or importance of "neutral point" is reduced. At transonic speeds the variation of control angle and trim force with speed, although important, is not affected by cg position. Instead of relating trim gradients to a cg margin, it is more useful to view variation of control for trim as a function of compressibility and ignore cg position.



## MANEUVERABILITY

REVISED 1 FEBRUARY 1974

## 4.1 MANEUVERING FLIGHT

The method used to analyze maneuvering flight will be to determine stick-fixed maneuver points ( $h_m$ ) and stick-free maneuver points ( $h_m^f$ ). It will be seen that these are analogous to their counterparts in static stability, the stick-fixed and stick-free neutral points. The maneuver points will also be defined in terms of the neutral points and the theory will help to predict which of these points will be critical as regards the aft center of gravity location. It will also be shown how the forward center of gravity is affected by the parameters that define the maneuver points.

Maneuvering flight will be analyzed much in the same manner used in determining a flight test relationship in longitudinal stability. For stick-fixed longitudinal stability, the flight test relationship was determined to be

$$\frac{d\delta_e}{dC_L} = - \frac{dC_m/dC_L}{C_{m\delta_e}} \quad (4.1)$$

This equation gave the static longitudinal stability of the aircraft in terms that could easily be measured in flight test.

In maneuvering flight, a similar stick-fixed equation relating to easily measurable flight test quantities is desirable. Where in longitudinal stability, the elevator deflection was related to lift coefficient or angle of attack, one may surmise that in maneuvering flight elevator deflection will relate to load factor  $n$ .

To determine this expression, one must refer to the aircraft's basic equations of motion. As in longitudinal stability, the six equations of motion are the basis for all analysis of aircraft stability and control. In maneuvering an aircraft the same equations will hold true, but one additional derivative will have to be added to the analysis. Recalling the pitching moment equation

$$M = \dot{q}I_y + pr(I_x - I_z) + (p^2 - r^2)I_{xz} \quad (4.2)$$

and the fact that in static stability analysis we have no roll rate, yaw rate, or pitch acceleration, equation 4.2 reduces to:

$$M = \dot{q}I_y = 0 \quad (4.3)$$

The variables that cause external pitching moments on an aircraft are infinite, i.e., speed brakes, canopy, elevator, etc. There are, however, five primary variables that we can consider.

$$M = f(V, \alpha, \dot{\alpha}, \delta_e, q) \quad (4.4)$$

If any or all of these variables change, there will be a change of total pitching moment that will equal the sum of the partial changes of all the variables. This is written as

$$\Delta M = \frac{\partial M}{\partial \alpha} \Delta \alpha + \frac{\partial M}{\partial \dot{\alpha}} \Delta \dot{\alpha} + \frac{\partial M}{\partial \delta_e} \Delta \delta_e + \frac{\partial M}{\partial V} \Delta V + \frac{\partial M}{\partial q} \Delta q \quad (4.5)$$

Since in maneuvering flight,  $\Delta V$  and  $\Delta \delta$  are zero, equation 4.5 becomes:

$$\Delta M = \frac{\partial M}{\partial \alpha} \Delta \alpha + \frac{\partial M}{\partial \delta_e} \Delta \delta_e + \frac{\partial M}{\partial q} \Delta q = 0 \quad (4.6)$$

and since  $M = q S c C_m$ , then

$$\frac{\partial M}{\partial \alpha} = q S c \frac{\partial C_m}{\partial \alpha} = q S c C_{m_\alpha} \quad (4.7)$$

$$\frac{\partial M}{\partial \delta_e} = q S c \frac{\partial C_m}{\partial \delta_e} = q S c C_{m_{\delta_e}} \quad (4.8)$$

$$\frac{\partial M}{\partial q} = q S c \frac{\partial C_m}{\partial q} \quad (4.9)$$

Substituting these values into equation 4.6, and multiplying by  $1/q S c$ ,

$$C_{m_\alpha} \Delta \alpha + C_{m_{\delta_e}} \Delta \delta_e + \frac{\partial C_m}{\partial q} \Delta q = 0 \quad (4.10)$$

The derivative  $\partial C_m / \partial q$  is carried instead of  $C_{m_q}$  since the compensating factor  $c/2V$  is not used at this time.

Solving for the change in elevator deflection  $\Delta \delta_e$ ,

$$\Delta \delta_e = \frac{-C_{m_\alpha} \Delta \alpha - \frac{\partial C_m}{\partial q} \Delta q}{C_{m_{\delta_e}}} \quad (4.11)$$

The analysis of equation 4.11 may be continued by substituting in values for  $\Delta \alpha$  and  $\Delta q$ . The final equation obtained should be in the form of some flight test relationship. Since maneuvering is related to load factor, the elevator deflection required to obtain dif-

ferent load factors will define the stick-fixed maneuver point. The immediate goal then is to determine the change in angle of attack,  $\Delta \alpha$ , and change in pitch rate,  $\Delta q$ , in terms of load factor  $n$ .

## 4.2 THE PULL UP MANEUVER

4.3 In the pull up maneuver, the change in angle of attack of the aircraft,  $\Delta \alpha$ , may be related to the lift coefficient of the aircraft. In the pull up with constant velocity, the angle of attack of the whole aircraft will be changed since the aircraft has to fly at a higher  $C_L$  to obtain the load factor required. The change in  $C_L$  required to maneuver at high load factors at a constant velocity comes from two sources: (1) load factor increase, (2) elevator deflection. Although often ignored because of its small value when compared to total  $C_L$ , the change in lift with elevator deflection  $C_{L_{\delta_e}} \Delta \delta_e$  will be carried along for a more general analysis.

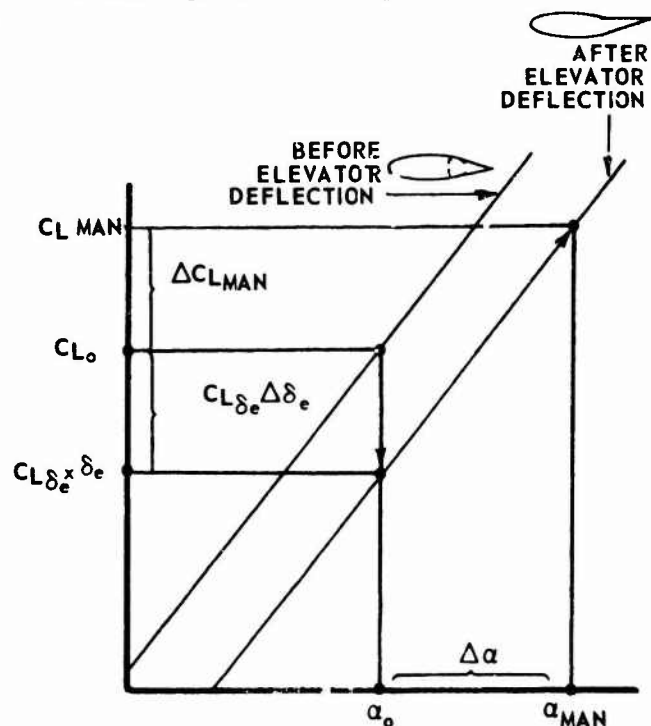


Figure 4.1 Lift Coefficient Versus Angle of Attack

Referring to figure 4.1, the aircraft is in equilibrium at some  $C_{L_0}$  corresponding to some  $\alpha_0$  before the elevator is deflected to initiate the pull up. If the elevator is considered as a flap, its deflection will affect the lift curve as follows. When the elevator is deflected upward, the lift curve shifts downward and does not change slope. This says that a certain amount of lift is initially lost when the elevator is deflected upward. The loss in lift because of elevator deflection is designated  $C_{L_{\delta_e}} \Delta \delta_e$ . The increase in down-loading on the tail or increase in negative lift on the horizontal stabilizer causes a moment on the aircraft which creates a nose up pitch rate. The aircraft continues to pitch upward and increase its angle of attack until it reaches a new  $C_L$  and an equilibrium load factor. In other words a pitch rate is initiated and  $\alpha$  increases until a maneuvering lift coefficient  $C_{L_{MAN}}$  is reached for the deflected elevator  $\delta_e$ . The change in angle of attack is  $\Delta \alpha$ . The change in  $C_L$  has come partially from the deflected elevator and mainly from the pitching maneuver. The change in  $C_L$  due to the maneuver is from  $C_{L_0}$  to  $C_{L_{MAN}}$ . Since it did not change the slope of the lift curve, if the change in lift caused by elevator deflection is included the expression for  $\Delta \alpha$  becomes:

$$C_L = a\alpha \quad (4.12)$$

$$\Delta C_L = a \Delta \alpha \quad (4.12a)$$

$$\Delta C_L = \Delta C_{L_{MAN}} - C_{L_{\delta_e}} \Delta \delta_e = a \Delta \alpha \quad (4.13)$$

$$\Delta \alpha = \frac{1}{a} [\Delta C_{L_{MAN}} - C_{L_{\delta_e}} \Delta \delta_e] \quad (4.14)$$

To put equation 4.14 in terms of load factor,  $\Delta C_{L_{MAN}}$  must be defined. This is the change in lift from the initial condition to the final maneuvering condition. This change can occur from one g flight to some other load factor or it can start at 2 or 3 g's and progress to some new load factor. If  $C_L$  is at one g then

$$C_L = \frac{W}{qS} \quad (4.15)$$

$$C_{L_0} = \frac{n_0 W}{qS} \quad n_0 - \text{initial load factor} \quad (4.16)$$

$$C_{L_{MAN}} = \frac{nW}{qS} \quad n - \text{final load factor} \quad (4.17)$$

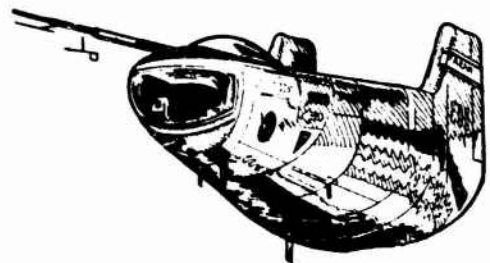
$$\begin{aligned} \Delta C_{L_{MAN}} &= C_{L_{MAN}} - C_{L_0} \\ &= \frac{nW}{qS} - \frac{n_0 W}{qS} = C_L (n - n_0) \end{aligned} \quad (4.18)$$

Finally substituting  $\Delta C_{L_{MAN}}$  into equation 4.14,

$$\Delta \alpha = \frac{1}{a} [C_L (n - n_0) - C_{L_{\delta_e}} \Delta \delta_e] \quad (4.19)$$

Equation 4.19 is now ready for substitution into equation 4.11.

An expression for  $\Delta q$  in equation 4.11 will be derived using the pull up maneuver analysis.



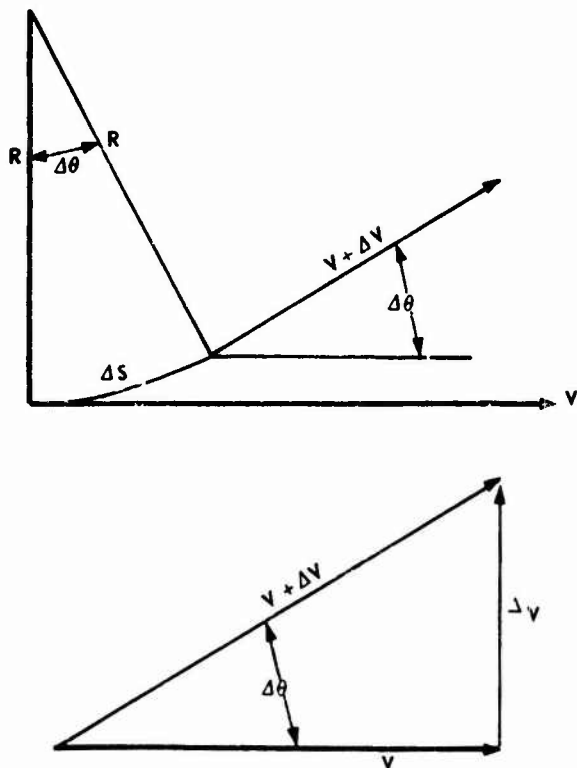


Figure 4.2 Curvilinear Motion

Referring to figure 4.2

$$\Delta \theta = \frac{\Delta S}{R} \quad (4.20)$$

$$\frac{d\theta}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta S}{\Delta t} \frac{1}{R} \quad (4.21)$$

$$\frac{d\theta}{dt} = \frac{V}{R} \quad (4.22)$$

From figure 4.2

$$\frac{\Delta V}{V} = \Delta \theta \quad (\text{small angles}) \quad (4.23)$$

$$\frac{d\theta}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta V}{\Delta t} \frac{1}{V} = \frac{1}{V} \frac{dV}{dt} \quad (4.24)$$

combining equations 4.24 and 4.22,

$$\frac{dV}{dt} = \frac{V^2}{R} \quad (4.25)$$

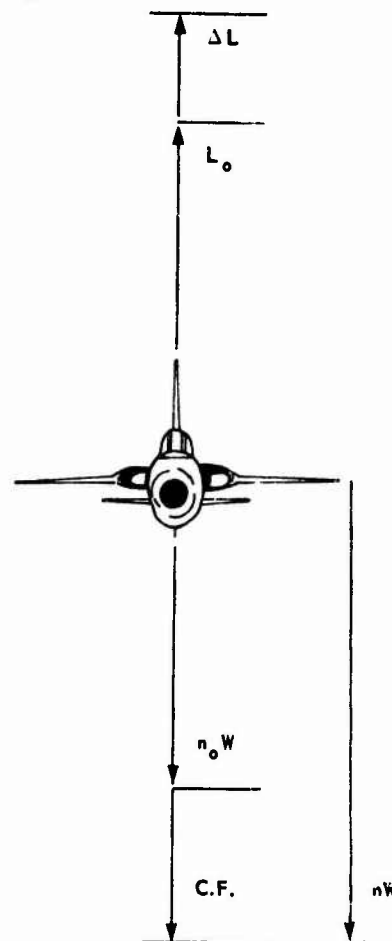


Figure 4.3 Wings Level Pull Up

From figure 4.3,

$$\Delta L = CF = nW - n_0W \quad (4.26)$$

Again the factor now indicates that the change may take place from any original load factor and is not limited to the straight and level flight condition. The centrifugal force that holds the aircraft in equilibrium can be expressed as:

$$CF = \frac{W}{g} a = \frac{W}{g} \frac{V^2}{R} \quad (4.27)$$

Therefore:

$$\Delta L = W(n - n_o) = \frac{W}{g} \frac{V^2}{R} \quad (4.28)$$

$$\frac{g}{V} (n - n_o) = \frac{V}{R} \quad (4.29)$$

Substituting from equation 4.22,

$$\Delta q \approx \frac{d\theta}{dt} = \frac{V}{R} = \frac{g}{V} (n - n_o) \quad (4.30)$$

Now equations 4.30 and 3.19 may be substituted into equation 4.11.

$$\Delta \delta_e = \frac{-C_{m\alpha} \frac{1}{a} [C_L (n - n_o) - C_{L\delta_e} \Delta \delta_e]}{C_{m\delta_e}} - \frac{\frac{\partial C_m}{\partial q} \frac{g}{V} (n - n_o)}{C_{m\delta_e}} \quad (4.31)$$

From longitudinal stability,

$$C_{m\alpha} = \frac{\partial C_m}{\partial C_L} \frac{\partial C_L}{\partial \alpha} = a (h - h_n) \quad (4.32)$$

Also to help further in reducing the equation to its simplest terms,

$$V^2 = \frac{2W}{\rho S C_L} \quad (4.33)$$

and

$$\frac{\partial C_m}{\partial q} = \frac{c}{2V} C_{mq} \quad (4.34)$$

Substituting equations 4.34, 4.33 and 4.22 into equation 4.31 and turning the algebra crank, results in,

$$\frac{\Delta \delta_e}{n - n_o} + \frac{a C_L}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} \left[ h - h_n + \rho \frac{Sc}{4m} C_{mq} \right] \quad (4.35)$$

Equation 4.35 is now in the form that will define the stick-fixed maneuver point for the pull up. The definition of the maneuver point ( $h_m$ ) is the cg position at which the elevator deflection per g goes to zero. Taking the limit of equation 4.35, where  $\Delta n$  is defined as  $(n - n_o)$ ,

$$\lim_{\Delta n \rightarrow 0} \frac{\Delta \delta_e}{\Delta n} = \frac{d\delta_e}{dn} \quad (4.36)$$

or

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} \left[ h - h_n + \rho \frac{Sc}{4m} C_{mq} \right] \quad (4.37)$$

Setting equation 4.37 equal to zero will give the cg position ( $h$ ) as the maneuver point ( $h_m$ ).

$$h_m = h_n - \rho \frac{Sc}{4m} C_{mq} \quad (4.38)$$

Solving equation 4.38 for  $h_n$  and substituting into equation 4.37,

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} (h - h_m) \quad (4.39)$$

where  $(h_m - h)$  is defined as the stick-fixed maneuver margin.

The significant points to be made about equation 4.39 are:

1. The derivative  $d\delta_e/dn$  varies with the maneuver margin. The more forward the cg, the more elevator will be required to obtain the limit load factor. That is, as the cg moves forward, more elevator deflection is necessary to obtain a given load factor.
2. The higher the  $C_L$ , the more elevator will be required to obtain the limit load factor. That is, at low speeds (high

$C_L$ ) more elevator deflection is necessary to obtain a given load factor than is required to obtain the same load factor at a higher speed (lower  $C_L$ ).

3. The derivative  $d\delta_e/dn$  should be linear with respect to cg at a constant  $C_L$ .

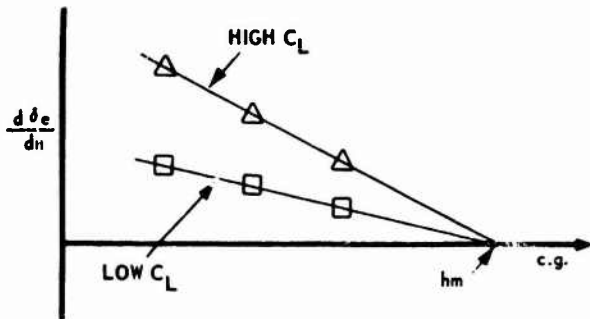


Figure 4.4  $\frac{d\delta_e}{dn}$  vs cg

Another approach to solving for the maneuver point ( $h_m$ ) is to return to the original stability equation.

$$\frac{dC_m}{dC_L} = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{LFus}} - \frac{a_t}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) \quad (4.40)$$

The effect of pitch damping on the aircraft stability will be determined and added to equation 4.40. Recalling the relationship:

$$\frac{dC_m}{dq} = \frac{c}{2V} C_{mq} \quad (4.41)$$

or

$$\Delta C_m = \frac{c}{2V} C_{mq} \Delta q \quad (4.42)$$

Substituting the value obtained for  $\Delta q$  from equation 4.30,

$$\Delta C_m = \frac{2g}{2V^2} C_{mq} (n - n_o) \quad (4.43)$$

Substituting the appropriate  $C_L$  expression for load factor,

$$\Delta C_m = \rho \frac{Sc}{4m} C_{mq} (C_{L_{MAN}} - C_{L_o}) \quad (4.44)$$

if

$$\Delta C_L = C_{L_{MAN}} - C_{L_o}, \quad \text{then}$$

$$\lim_{\Delta C_L \rightarrow 0} \frac{\Delta C_m}{\Delta C_L} = \frac{dC_m}{dC_L} = \rho \frac{Sc}{4m} C_{mq} \quad (4.45)$$

Pitch Damping

This term may now be added to equation 4.45. If the sign of  $C_{mq}$  is negative, then the term is a stabilizing contribution to the stability equation.  $C_{mq}$  will be analyzed further.

$$\frac{dC_m}{dC_L} = h - \frac{X_{ac}}{c} + \frac{dC_m}{dC_{LFus}} - \frac{a_t}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) + \rho \frac{Sc}{4m} C_{mq} \quad (4.46)$$

The maneuver point is found by setting  $dC_m/dC_L$  equal to zero and solving for the cg position where this occurs.

$$h_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{LFus}} + \frac{a_t}{a_w} V_H \eta_T \left(1 - \frac{d\epsilon}{d\alpha}\right) - \rho \frac{Sc}{4m} C_{mq} \quad (4.47)$$

The first three terms on the right side of equation 4.47 may be identified as the expression for the neutral point  $h_n$ . If this substitution is made in equation 4.47, equation 4.38 is again obtained.

$$h_m = h_n - \rho \frac{Sc}{4m} C_{mq} \quad (4.48)$$

The derivative  $C_{mq}$  found in equation 4.37, 4.38, and 4.46 needs to be examined before proceeding with further discussion.

The damping that comes from the pitch rate established in a pull up, comes from the wing, tail, and fuselage components. The tail is the largest contributor to the pitch damping because of the long moment arm. For this reason it is usually used to derive the value of  $C_{mq}$ . Sometimes an empirical value is added to account for the rest of the aircraft but more often than not, the value for the tail alone is used to estimate the derivative. The effect of the tail may be calculated in the following manner.

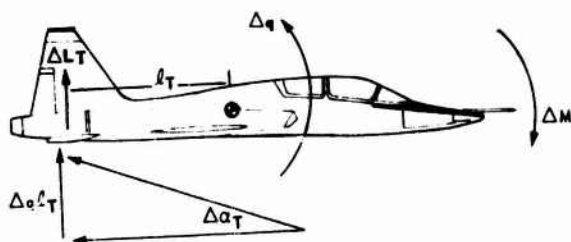


Figure 4.5 Pitch Damping

The pitching moment effect on the aircraft from the downward moving horizontal stabilizer is:

$$\Delta M = -l_T \Delta L_T = q_w S_w c_w \Delta C_m \quad (4.49)$$

where

$$\Delta L_T = q_T S_T \Delta C_{L_T} \quad (4.50)$$

Solving for  $\Delta C_m$ ,

$$\Delta C_m = - \frac{q_T l_T S_T}{q_w c_w S_w} \Delta C_{L_T} \quad (4.51)$$

The combination  $l_T S_T / c_w S_w$  can be recognized as the tail volume coefficient  $V_H$ . The term  $q_T / q_w$  is

referred to as the tail efficiency factor  $\eta_T$ .

Equation 4.51 may then be written:

$$\Delta C_m = - V_H \eta_T \Delta C_{L_T} \quad (4.52)$$

which can be further refined to:

$$\Delta C_m = - V_H \eta_T a_T \Delta \alpha_T \quad (4.53)$$

From figure 4.5, the change in angle of attack at the tail caused by the pitch rate will be:

$$\Delta \alpha_T = \tan^{-1} \frac{\Delta q l_T}{V} \approx \Delta q \frac{l_T}{V} \quad (4.54)$$

Substituting equation 4.54 into 4.53

$$\Delta C_m = - a_T V_H \eta_T \frac{l_T}{V} \Delta q \quad (4.55)$$

Taking the limit of equation 4.55 gives

$$\frac{\partial C_m}{\partial q} = - a_T V_H \eta_T \frac{l_T}{V} \quad (4.56)$$

Equation 4.56 shows that the damping expression  $dC_m/dq$  is a function of airspeed, i.e., this term is greater at lower speeds.

Solving for  $C_{mq}$ ,

$$C_{mq} = \frac{2V}{c} \frac{\partial C_m}{\partial q} = - 2a_T V_H \eta_T \frac{l_T}{c} \quad (4.57)$$

The damping derivative is not a function of airspeed but rather a value determined by design considerations only (subsonic flight). The damping in pitch derivative may be increased by increasing  $S_T$  or  $l_T$ .

When this value for  $C_{mq}$  is substituted into equation 4.48,

$$h_m = h_n + \rho \frac{S a_T \eta_T \ell_T}{2m} v_H \quad (4.58)$$

The following conclusions are apparent from equation 4.58.

1. The maneuver point should always be behind the neutral point. This is verified since the addition of a pitch rate increases the stability ( $C_{mq}$  is negative in equation 4.46) of the aircraft. Therefore, the stability margin should increase.
2. Aircraft geometry is influential in locating the maneuver point aft of the neutral point.
3. As altitude increases, the distance between the neutral point and maneuver point decreases.
4. As weight decreases at any given altitude, the maneuver point moves further behind the neutral point and the maneuver stability margin increases.
5. The largest variation between maneuver point and neutral point occurs with a light aircraft flying at sea level.

### ■ 4.3 AIRCRAFT BENDING

Before the pull up analysis is completed, one more subject should be covered. One of the assumptions made early in stability was that the aircraft was a rigid body. It is a well known fact that all aircraft bend when a load is applied. The bigger the aircraft, the more they bend. The effect on the aircraft bending is shown in figures 4.6a and 4.6b.

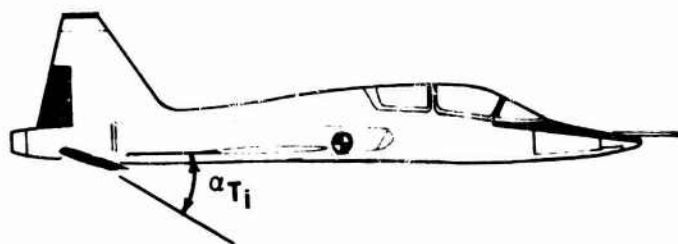


Figure 4.6a Rigid Aircraft Under High Load Factor

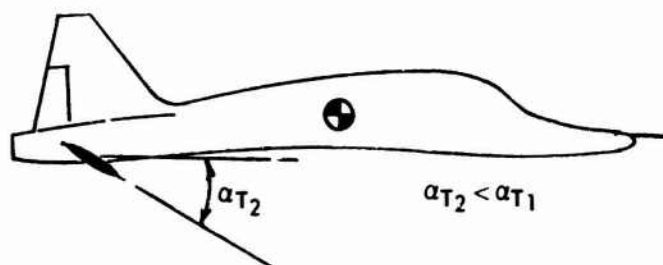


Figure 4.6b Non-Rigid Aircraft under High Load Factor

The angle of attack of the tail is approximately the same as the angle of attack of the wing with the exception of downwash, incidence, etc., for a particular elevator deflection. As the non-rigid aircraft bends, the angle of attack  $\alpha_T$  of the horizontal stabilizer decreases. In order to keep the aircraft at the same overall angle of attack, the original angle of attack of the tail must be reestablished. This requires an increase in the elevator (slab) deflection or a  $\Delta\delta_e$  to reestablish the necessary  $\alpha_T$  and to maintain the required maneuvering  $C_L$ . This additional elevator requirement under aircraft bending gives an apparent increase in the maneuvering stability of the aircraft or an additional  $\Delta\delta_e$  per load factor.

### ■ 4.4 THE TURN MANEUVERING

The subject of maneuvering in pull ups has been thoroughly discussed and while it is the easiest method for a test pilot to perform, it is also the most time consuming.

Therefore, most maneuvering data is collected by turning. There are several methods used to collect data in a turn and these are discussed in the flight test portion of this text.

In order to analyze the maneuvering turn, equation 4.11 is recalled:

$$\Delta \delta_e = - \frac{C_{m_\alpha} \Delta \alpha - \partial C_m / \partial q \Delta q}{C_{m_{\delta_e}}} \quad (4.59)$$

The expression for  $\Delta \alpha$  in equation 3.19, derived for the pullup maneuver, is also applicable to the turning maneuver.

$$\Delta \alpha = \frac{1}{a} [C_L (n - n_0) - C_{L_{\delta_e}} \Delta \delta_e] \quad (4.60)$$

Such is not the case for the  $\Delta q$  expression in equation 4.59. Another expression for  $\Delta q$  pertaining to the turn maneuver must be developed.

Referring to figure 4.7, the left vector will be statically balanced by the weight and centrifugal force. One component ( $L \cos \phi$ ) balances the weight and the other ( $L \sin \phi$ ) balances the centrifugal force.

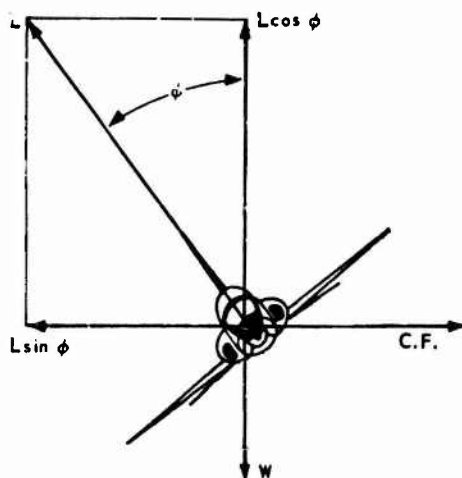


Figure 4.7a

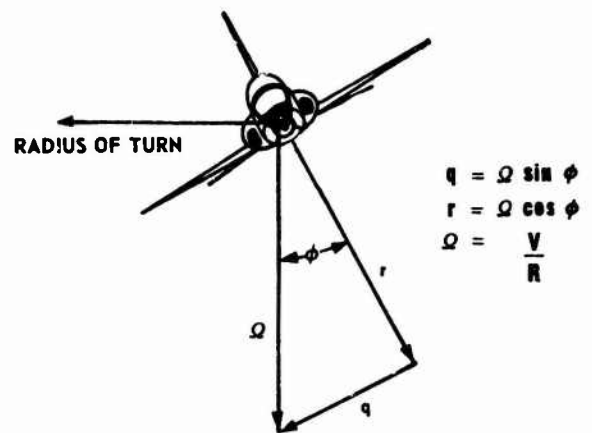


Figure 4.7b Aircraft in the Turn Maneuver

$$L \sin \phi = CF = \frac{W}{g} \frac{V^2}{R} \quad (4.61)$$

$$L \cos \phi = W \quad (4.62)$$

$$n = L/W = \frac{1}{\cos \phi} \quad (4.63)$$

Now dividing 4.61 by 4.62 and rearranging terms:

$$\frac{V}{R} = \frac{g}{V} \frac{\sin \phi}{\cos \phi} \quad (4.64)$$

Referring to figure 4.7 where pitch rate is represented by a vector along the wings and yaw rate a vector vertically through the center of gravity, the following relationships can be derived.

$$\Omega = \frac{V}{R} \quad (4.65)$$

$$q = \Omega \sin \phi \quad (4.66)$$

$$q = \frac{V}{R} \sin \phi \quad (4.67)$$

Substituting 4.64 into 4.67,

$$q = \frac{g}{V} \frac{\sin^2 \theta}{\cos \theta} \quad (4.68)$$

$$q = \frac{g}{V} \frac{1 - \cos^2 \theta}{\cos \theta} \quad (4.69)$$

$$q = \frac{g}{V} \left( \frac{1}{\cos \theta} - \cos \theta \right) \quad (4.70)$$

$$q = \frac{g}{V} \left( n - \frac{1}{n} \right) \quad (4.71)$$

When maneuvering from initial conditions of  $n_0$  to  $n$ , the  $\Delta q$  equation becomes,

$$\Delta q = q - q_0 = \frac{g}{V} \left( n - \frac{1}{n} \right) - \frac{g}{V} \left( n_0 - \frac{1}{n_0} \right) \quad (4.72)$$

$$\Delta q = \frac{g}{V} \left( n - n_0 \right) \left( 1 + \frac{1}{nn_0} \right) \quad (4.73)$$

The general expression for  $\Delta q$  in equation 4.73 and the value for  $\Delta \alpha$  in equation 4.60 may now be substituted into equation 4.59 to determine  $\Delta \delta_e$

$$\Delta \delta_e = \frac{-C_{m\alpha} \frac{1}{a} \left[ C_L (n - n_0) - C_{L\delta_e} \Delta \delta_e \right]}{C_{m\delta_e}} \quad (4.74)$$

$$- \frac{\partial C_m}{\partial q} \frac{g}{V} \left( n - n_0 \right) \left( 1 + \frac{1}{nn_0} \right)$$

Substituting

$$\begin{aligned} \frac{\partial C_m}{\partial q} &= \frac{c}{2V} C_{mq} \\ \left( C_{m\delta_e} - \frac{C_{m\alpha} C_{L\delta_e}}{a} \right) \Delta \delta_e &= \\ &- \frac{C_{m\alpha} C_L}{a} (n - n_0) \\ &- C_{mq} \frac{cg}{2V^2} (n - n_0) \left( 1 + \frac{1}{nn_0} \right) \end{aligned} \quad (4.75)$$

$$\Delta \delta_e = \frac{C_{m\alpha} C_L (n - n_0) + C_{mq} a \frac{cg}{2V^2} (n - n_0) \left( 1 + \frac{1}{nn_0} \right)}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} \quad (4.76)$$

Now

$$C_{m\alpha} = a (h - h_n) \text{ and } V^2 = \frac{2W}{C_L \rho S}$$

$$\frac{\Delta \delta_e}{(n - n_0)} = \frac{a C_L}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} \cdot \left[ (h - h_n) + C_{mq} \frac{\rho S c}{4m} \left( 1 + \frac{1}{nn_0} \right) \right] \quad (4.77)$$

Taking the limit of  $\Delta \delta_e / \Delta n$  in equation 4.77 and,

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} \cdot \left[ h - h_n + \frac{\rho S c}{4m} C_{mq} \left( 1 + \frac{1}{n^2} \right) \right] \quad (4.78)$$

The maneuver point is determined by setting  $d\delta_e/dn$  equal to zero and solving for the cg position at this point.

$$h_m = h_n - \frac{\rho S c}{4m} C_{mq} \left( 1 + \frac{1}{n^2} \right) \quad (4.79)$$

The maneuver point in a turn differs from the pullup by the factor  $(1 + 1/n^2)$ . This means that at low load factors the turn and pullup maneuver points will be very nearly the same. If equation (4.79) is solved for  $h_n$  and substituted back into equation 4.76, the result is:

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} (h - h_m) \quad (4.80)$$

The conclusion that  $d\delta_e/dn$  is the same for both pullup and turn would be untrue since  $h_m$  in equation 4.80 for turns (includes the factor  $(1 + 1/n^2)$  is different from the  $h_m$  found for the pullup maneuver. The same conclusions reached for 4.39 and 4.58 apply to 4.80 and 4.81 as well.

$$h_m = \frac{h_n + \rho S a_T \eta_T l_T}{2m} V_H (1 + 1/n^2) \quad (4.81)$$

#### ■ 4.5 RECAPITULATION

Before looking further into the stick-free maneuverability case, it would be well to review the development in the preceding paragraphs and relate it to the results of chapter 3.

The basic approach to longitudinal stability was centered around finding a value for  $dC_m/dC_L$ . It was found that a negative value for this derivative meant that the aircraft was statically stable. The derivative was analyzed for the stick-fixed case first and then the stick-free case. The cg position where this derivative was zero, was defined as the neutral point. Static margin was defined as the difference between the neutral point and the cg location. The stick-free case was determined by:

$$\left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Free Aircraft}} = \left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Fixed Aircraft}} + \Delta \left. \frac{dC_m}{dC_L} \right|_{\text{Effect of Free Elevator}} \quad (4.82)$$

The free elevator case was merely the basic stability of the aircraft with the effect of freeing the elevator added to it.

When the maneuvering case was introduced, it was shown that there was a new derivative to be discussed but the basic stability of the aircraft would not change - only the effect of pitch rate was added to it.

$$\left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Fixed Aircraft Pitching}} = \left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Fixed Aircraft}} + \Delta \left. \frac{dC_m}{dC_L} \right|_{\text{Effect of The Pitch Rate}} \quad (4.83)$$

For the stick-free case, the following must be true,

$$\left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Free Aircraft Pitching}} = \left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Fixed Aircraft}} + \Delta \left. \frac{dC_m}{dC_L} \right|_{\text{Effect of Free Elevator}} + \Delta \left. \frac{dC_m}{dC_L} \right|_{\text{Effect of Pitch Rate}} \quad (4.84)$$

$$\left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Free Aircraft Pitching}} = \left. \frac{dC_m}{dC_L} \right|_{\text{Stick-Free Aircraft}} + \Delta \left. \frac{dC_m}{dC_L} \right|_{\text{Effect of Pitch Rate}} \quad (4.85)$$

#### ■ 4.6 STICK-FREE MANEUVERING

The first analysis of stick-free maneuvering requires a review of longitudinal stability. It was determined in chapter 3 that the effect of freeing the elevator was to multiply the tail term by the free elevator factor  $F$  which equaled  $(1 - T Ch_\alpha/Ch_\delta)$ . Consequently, to free the elevator in the maneuvering case and find the stick-free maneuver point, the tail effect of stick-fixed maneuvering must be multiplied by this free elevator factor. Recalling equation 4.47 from the stick-fixed maneuvering discussion,

$$h_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H \eta_T \cdot (1 - \frac{d\epsilon}{d\alpha}) - \rho \frac{Sc}{4m} C_{mq} \quad (4.86)$$

Multiplying the tail terms by  $F$ ,

$$h'_m = \frac{X_{ac}}{c} - \frac{dC_m}{dC_{L_{Fus}}} + \frac{a_t}{a_w} V_H \eta_T \cdot (1 - \frac{d\epsilon}{d\alpha}) F - \rho \frac{Sc}{4m} C_{mq} F \quad (4.87)$$

The first three terms on the right are the expression for stick-free neutral point,  $h'_n$ .

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{mq} F \quad (4.88)$$

This is the stick free maneuver point in terms of the stick-free neutral point for the pullup case. It may be extended to the turn case by using the term for the pitch rate of the tail in a turn.

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{mq} F \left(1 + \frac{1}{2}\right) \quad (4.89)$$

These equations do not give any flight test relationship and so it is necessary to derive this from stick forces, as was done in longitudinal static stability. The method used will be to relate the stick force-per-g to the stick-free maneuver point since stick forces can be related to the freeing of the elevator. Starting with the relationship of stick force, gearing, and hinge moments that was derived in chapter 3,

$$F_s = -GH_e \quad (4.90)$$

$$H_e = q S_e c_e C_{h_e} \quad (4.91)$$

$$F_s = -Gq S_e c_e C_{h_e} \quad (4.92)$$

The change in stick force for a change in load factor becomes,

$$\frac{\Delta F_s}{\Delta n} = -Gq S_e c_e \frac{\Delta C_{h_e}}{\Delta n} \quad (4.93)$$

where

$$\Delta C_{h_e} = C_{h_{\alpha_T}} \Delta \alpha_T + C_{h_{\delta_e}} \Delta \delta_e \quad (4.94)$$

### Stick-Free Pullup Maneuver:

$\Delta C_{h_e}$  must be written in terms of load factor and substituted back into equation 4.93. This will require defining  $\Delta \alpha_T$  and  $\Delta \delta_e$  in terms of load factor. The change in angle of attack of the tail comes partly from the change in angle of attack of the wing due to downwash and partly from the pitch rate.

$$\Delta \alpha_T = \Delta \alpha_W \left(1 - \frac{d\epsilon}{d\alpha}\right) + \Delta q \frac{\ell_T}{V} \quad (4.95)$$

where  $\Delta \alpha_W + \Delta q$  in the above equation are

$$\Delta \alpha_W = \frac{1}{a} \left[ C_L (n - n_0) - C_{L_{\delta_e}} \Delta \delta_e \right] \quad (4.96)$$

$$\Delta q = \frac{g}{V} (n - n_0) \quad (4.97)$$

$$\Delta \delta_e = \frac{a' L}{C_{m_{\alpha}} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m) (n - n_0) \quad (4.98)$$

If the equations above are substituted into 4.94, the results would be cumbersome at best. To simplify things  $C_{L_{\delta_e}}$  will be assumed small enough to ignore. (Reasonable assumption since total change in lift of the aircraft when the elevator is deflected is small.) The above equations simplify to:

$$\Delta \alpha_T = \frac{C_L}{a} \left(1 - \frac{d\epsilon}{d\alpha}\right) \cdot (n - n_0) + \frac{g}{V^2} \ell_T (n - n_0) \quad (4.99)$$

$$\Delta \delta_e = -\frac{C_L}{C_{m_{\delta_e}}} (h - h_m) (n - n_0) \quad (4.100)$$

Substituting equations 4.99 and 4.100 into 4.94,

$$\frac{\Delta C_{h_e}}{n - n_0} = C_{h_\alpha} \frac{C_L}{a} \left(1 - \frac{d\epsilon}{d\alpha}\right) + C_{h_\alpha} g \frac{\ell_T}{v^2} - C_{h_\delta} \frac{C_L}{C_{m_{\delta_e}}} (h - h_m) \quad (4.101)$$

Substituting  $v^2 = W/\rho S C_L$  and  $C_{m_\delta} = -a_e V_H$  and isolating the maneuver margin  $(h - h_m)$  by factoring out  $(-C_{h_\delta} C_L/C_{m_{\delta_e}})$ , the result is:

$$\frac{\Delta C_h}{n - n_0} = - \frac{C_{h_\delta} C_L}{C_{m_{\delta_e}}} \left[ \frac{C_{h_\alpha}}{C_{h_\delta} a_W} \left(1 - \frac{d\epsilon}{d\alpha}\right) a_e V_H + \frac{C_{h_\alpha}}{C_{h_\delta}} \rho \frac{\ell_T}{2m} S a_e V_H + h - h_m \right] \quad (4.102)$$

From longitudinal stability,

$$h_n - h'_n = \frac{C_{h_\alpha}}{C_{h_\delta}} \frac{a_e}{a_W} V_H \left(1 - \frac{d\epsilon}{d\alpha}\right) \quad (4.103)$$

and if the second term in the parenthesis is multiplied by

$$\frac{-2ca_T}{-2ca_T} \text{ and knowing that } C_{m_q} = -2a_T \frac{V_H}{c} \ell_T \quad (4.104)$$

$$\tau = a_e/a_T \quad (4.105)$$

$$F = 1 - \tau \frac{C_{h_\alpha}}{C_{h_\delta}} \quad (4.106)$$

The second term becomes:

$$(F-1) \rho \frac{Sc}{4m} C_{m_q} \quad (4.107)$$

Rewriting equation 4.102,

$$\frac{\Delta C_h}{n - n_0} = + \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \left[ h'_n - h_n + (1 - F) \rho \frac{Sc}{4m} C_{m_q} - h + h_m \right] \quad (4.108)$$

but

$$h_m = h_n - \rho \frac{Sc}{4m} C_{m_q} \quad (4.109)$$

Therefore:

$$\frac{\Delta C_h}{n - n_0} = - \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \left[ h - h'_n + \rho \frac{Sc}{4m} C_{m_q} F \right] \quad (4.110)$$

Substituting equation 4.110 back into 4.93 and taking the limit

$$\frac{dF_s}{dn} = G 1/2 \rho v^2 S_e c_e \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \left[ h - h'_n + \rho \frac{Sc}{4m} C_{m_q} F \right] \quad (4.111)$$

Defining the stick-free maneuver point as the cg position where  $dF_s/dn$  is equal to zero,

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{m_q} F \quad (4.112)$$

which is the same equation as 4.88 previously derived. Equation 4.111 may be written,

$$\frac{dF_s}{dn} = G 1/2 \rho v^2 S_e c_e \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_L \left[ h - h'_m \right] \quad (4.113)$$

Equation 4.113 may be rearranged if the following substitutions are made.

$$C_{m\delta_e} = -a_e V_H = a_T \frac{\partial \alpha_T}{\partial \delta_e} V_H$$

$$= -C_{L\delta_e} \frac{l_T S_T}{c_w S_w} \quad (4.114)$$

$$C_L = \frac{2W}{\rho V^2 S} \quad (4.115)$$

The stick-force-per-g equation becomes:

$$\frac{dF_s}{dn} = -G (S_e c_e W C_{h\delta}) \frac{c_w}{l_T S_T C_{L\delta_e}}$$

$$\left[ h - h'_m \right] \quad (4.116)$$

#### Stick-Free Turn Maneuver:

The procedure used for determining the  $dF_s/dn$  equation and an expression for the stick-free maneuver point for the turning maneuver is practically identical to the pullup case. For the turn condition  $\Delta q$  is now,

$$\Delta q = \frac{g}{V} (n - n_o) \left( 1 + \frac{1}{nn_o} \right) \quad (4.117)$$

The change in angle of attack of the tail,  $\Delta \alpha_T$  and  $\Delta \delta_e$  become

$$\Delta \alpha_T = \frac{C_L}{a} (n - n_o) \left( 1 - \frac{d\epsilon}{d\alpha} \right) +$$

$$\frac{g}{V^2} \frac{l_T}{2} (n - n_o) \left( 1 + \frac{1}{nn_o} \right) \quad (4.118)$$

$$\Delta \delta_e = \frac{-C_{L\delta_e}}{C_{m\delta_e}} \left[ (h - h'_m) (n - n_o) + \right.$$

$$\left. \rho \frac{Sc}{4m} C_{mq} (n - n_o) \left( 1 + \frac{1}{nn_o} \right) \right] \quad (4.119)$$

Substituting equations 4.118 and 4.119 into equation 4.94 and performing the same factoring and substitutions as in the pullup case;

then,

$$\frac{\Delta C_h}{n - n_o} = - \frac{C_{h\delta} C_L}{C_{m\delta_e}}$$

$$\left[ h - h'_n + \rho \frac{Sc}{4m} C_{mq} F \left( 1 + 1/n^2 \right) \right] \quad (4.120)$$

Substituting 4.120 into 4.93

$$\frac{dF_s}{dn} = G \frac{1}{2} \rho V^2 S_e c_e \frac{C_{h\delta} C_L}{C_{m\delta_e}}$$

$$\left[ h - h'_n + \rho \frac{Sc}{4m} C_{mq} F \left( 1 + \frac{1}{n^2} \right) \right] \quad (4.121)$$

And solving for the stick-free maneuver point,

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{mq} F \left( 1 + 1/n^2 \right) \quad (4.122)$$

Further substitution puts equation 4.121 into the following form:

$$\frac{dF_s}{dn} = -G (S_e c_e W C_{h\delta}) \frac{c_w}{l_T S_T C_{L\delta_e}}$$

$$\left[ h - h'_m \right] \quad (4.123)$$

Again, the turning stick-force-per-g equation 4.123 appears identical to the stick-free pullup equation. However, the expression for the maneuver point  $h'_m$  is different.

The term in the first parenthesis represents the hinge moment of the elevator. The second term is the elevator power and the last term is the negative value of the

stick-free maneuver margin. The following conclusions are drawn from this equation.

1. The stick-force-per-g appears to vary directly with the weight. However, weight also appears inversely in  $h'_m$ . Therefore, the full effect of weight cannot be truly analyzed since one effect could cancel the other.
2. Since airspeed does not appear in the equation, the stick-force-per-g will be the same at all airspeeds for a fixed cg.

From equation 3.112, come the following conclusions.

1. The difference between the stick-fixed and stick-free maneuver point is a function of the free elevator factor,  $F$ .
2. The stick-free maneuver point,  $h'_m$ , varies directly with altitude, becoming closer to the stick-free neutral point, the higher the aircraft flies.
3. The location of the stick-free maneuver point is of academic interest only since it occurs at the point where  $dF_s/dn = 0$ . It is difficult to fly an aircraft with this type gradient. Consequently, military specifications limit the minimum value of  $dF_s/dn$  to three pounds per g.
4. The forward cg is limited by stick force per g. The maximum value is limited by the type aircraft (bomber, fighter, or trainer); i.e., heavier gradients in bomber type and lighter ones in fighters.

#### 4.7 EFFECT OF BOBWEIGHTS AND SPRINGS

The effect of bobweights and springs on the stick-free maneuver point and stick-force gradients is of interest. The result of adding a spring or a bobweight to the control system adds an incremental force to the system. The effect of the spring is different from the effect of the bobweight. The spring exerts a constant force on the stick no matter what load factor is applied. The bobweight exerts a force on the stick proportional to the load factor.

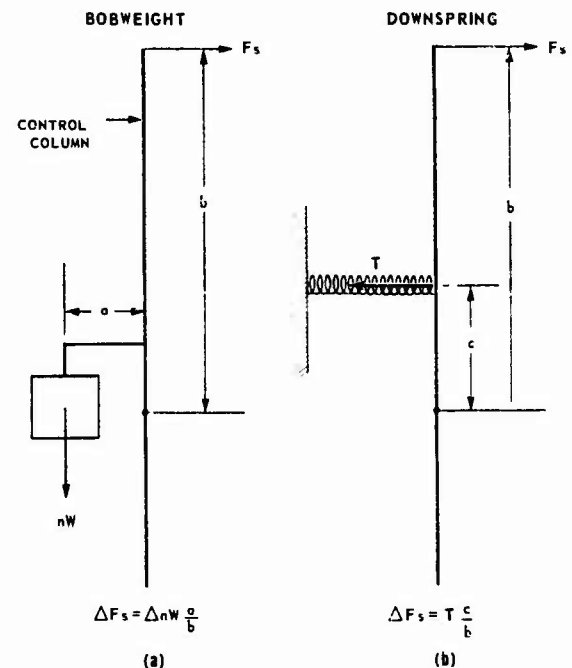


Figure 4.8 Bobweight and Downspring

The force increment for the downspring and bobweight are:

$$\Delta F_s = - \frac{G S_e c_e W Ch_0 c_w}{\ell_T S_i c_{L_{\delta_e}}} \quad \text{Spring}$$

$$(h - h'_m)(n - n_0) + T \frac{c}{b} \quad (4.124)$$

$$\Delta F_s = - \frac{G S_e c_e W C_{h\delta}}{\ell_T S_T C_{L\delta_e}} c_w (h - h'_m)(n - n_0) + W \frac{a}{b} (n - n_0) \quad (4.125)$$

Bobweight

When the derivative is taken with respect to load factor, the effect on  $dF_s/dn$  of the spring is zero. The stick force gradient is not affected by the spring nor is the stick-free maneuver point changed.

$$\frac{dF_s}{dn} = - \frac{G S_e c_e W C_{h\delta}}{\ell_T S_T C_{L\delta_e}} c_w (h - h'_m) \quad (4.126)$$

Spring

For the bobweight, the stick force gradient  $dF_s/dn$  becomes:

$$\frac{dF_s}{dn} = - \frac{G S_e c_e W C_{h\delta}}{\ell_T S_T C_{L\delta_e}} c_w (h - h'_m) + W \frac{a}{b} \quad (4.127)$$

Bobweight

Consequently, the addition of the bobweight (positive) increases the stick force gradient, moves the stick-free maneuver point aft, and shifts the allowable cg spread aft (the minimum and maximum cg positions as specified by force gradients are moved aft). See figure 4.9.

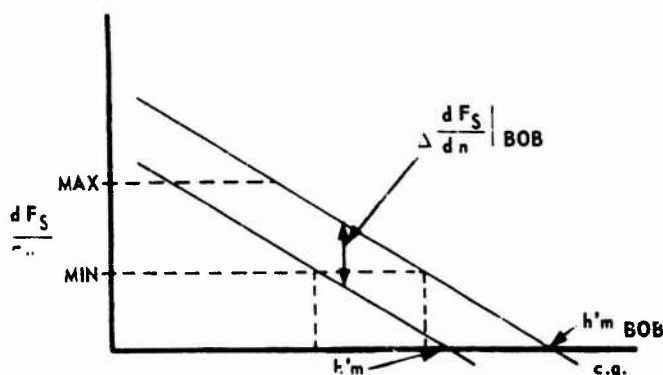


Figure 4.9 Effects of Adding a Bobweight

## 4.8 AERODYNAMIC BALANCING

Aerodynamic balancing is used to affect the stick force gradient and stick-free maneuver point. Aerodynamic balancing or varying values of  $C_{h\alpha}$  and  $C_{h\delta}$  affects the following stick-free equations.

$$\frac{dF_s}{dn} = - \frac{G S_e c_e W C_{h\delta}}{\ell_T S_T C_{L\delta_e}} c_w (h - h'_m) \quad (4.128)$$

$$h'_m = h'_n - \rho \frac{S c}{4m} C_{mq} F \quad (4.129)$$

$$F = 1 - \tau \frac{C_{h\alpha}}{C_{h\delta}} \quad (4.130)$$

Decreasing  $C_{h\delta}$  and/or increasing  $C_{h\alpha}$  by using two such aerodynamic balanced devices as an overhang balance or a lagging balance tab, does the following:

1. The free elevator factor,  $F$ , decreases.
2. The stick-free maneuver point  $h'_m$  moves forward.
3. The maneuver margin term  $(h - h'_m)$  decreases.
4. The stick force gradient decreases.
5. The forward and aft cg limits move forward.

Increasing  $C_{h\delta}$  and/or decreasing  $C_{h\alpha}$  by using a convex trailing edge or a leading balance tab does the following:

1. The free elevator factor,  $F$ , increases.
2. The stick-free maneuver point  $h'_m$  moves aft.

3. The maneuver margin term  $(h - h'_m)$  increases.
4. The stick force gradient increases.
5. The forward and aft cg limits move aft.

#### 4.9 cg RESTRICTIONS

The restrictions on the aircraft's center of gravity location may be examined by referring to the mean aerodynamic chord in figure 4.10.

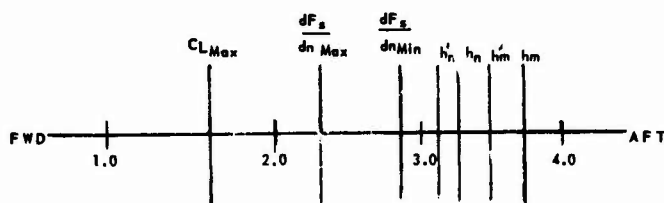


Figure 4.10 Restrictions to Center of Gravity Locations

The forward cg travel is normally limited by:

1. Maximum stick-force-per-g gradient -  $dF_s/dn$ .

or

2. Elevator required to land at  $C_{LMAX}$ .

The aft cg travel is normally limited by:

1. Minimum stick-force-per-g -  $dF_s/dn$ .

or

2. Stick-free neutral point -  $h'_n$ .

Additional considerations:

1. Freeing the elevator causes a destabilizing moment that locates the stick-free neutral

and maneuver points ahead of their respective stick-fixed points.

2. The stick-free neutral point,  $h'_n$ , can be moved aft artificially with a downspring. The stick-free maneuver point,  $h'_m$ , can be moved aft with a bobweight but not a downspring.
3. The desired aft cg location may be unsatisfactory because it lies aft of the cg position giving minimum stick force gradient. The requirement for bobweight or a particular aerodynamic balancing would exist in order to shift the cg for minimum stick force gradient aft of the desired aft cg position.

The equations which pertain to maneuvering flight are repeated below:

#### Pull Ups, Stick-Fixed

$$h_m = h_n - \rho \frac{Sc}{4m} C_{mq} \quad (4.131)$$

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m_x} C_{L_{\delta_e}} - C_{m_{\delta_e}} a} (h - h_m) \quad (4.132)$$

#### Pull Ups, Stick-Free

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{mq} F \quad (4.133)$$

$$\frac{dF_s}{dn} = - \frac{G S_e c_e W C_{h\delta} c_w}{l_T S_T C_{L_{\delta_e}}} (h - h'_m) \quad (4.134)$$

$$\frac{dF_s}{dn} = - \frac{G S_e c_e W C_{h\delta}}{\ell_T S_T C_{L\delta_e}} c_w (h - h'_m) + W \frac{a}{b}$$

Bobweight (4.135)

#### Turns, Stick-Free

$$h'_m = h'_n - \rho \frac{Sc}{4m} C_{mq} (1 + 1/n^2) \quad (4.138)$$

#### Turns, Stick-Fixed

$$h_m = h_n - \rho \frac{Sc}{4m} C_{mq} (1 + \frac{1}{n^2}) \quad (4.136)$$

$$\frac{dF_s}{dn} = - \frac{G S_e c_e W C_{h\delta}}{\ell_T S_T C_{L\delta_e}} c_w (h - h'_m) \quad (4.139)$$

$$\frac{d\delta_e}{dn} = \frac{a C_L}{C_{m\alpha} C_{L\delta_e} - C_{m\delta_e} a} (h - h_m) \quad (4.137)$$



# CHAPTER LATERAL-DIRECTIONAL STATIC STABILITY

# 5

(REVISED FEBRUARY 1974)

## 5.1 INTRODUCTION

An analysis of the equations of aircraft motion leads to the following mathematical description of aircraft lateral-directional motion:

$$F_y = m\dot{v} + mru - p\omega m \quad (5.1)$$

$$G_x = \dot{p}I_x + qr(I_z - I_y) - (\dot{r} + pq)I_{xz} \quad (5.2)$$

$$G_z = \dot{r}I_z + pq(I_y - I_x) + (qr - \dot{p})I_{xz} \quad (5.3)$$

The right side of the equation represents the response of an aircraft to applied forces and moments. The forces and moments are expressed on the left side of the equation in terms of stability derivatives and small perturbations. As in "Long-Stat", an analysis of the lateral-directional static stability need only concern itself with the values of these derivatives. Further analysis of the aircraft equations of motion reveals the left side of the foregoing equations to be composed primarily of contributions from aerodynamic forces and moments, direct thrust, gravity, and gyroscopic moments. Of these, only the aerodynamic forces and moments ( $Y, \dots$ ) will be analyzed because the other sources are usually eliminated through proper design.

It has been shown in Chapter 1 that when operating under a small disturbance assumption, aircraft lateral-directional motion can be considered independent of longitudinal motion and that it can be considered as a function of the following variables:

$$(Y, \mathcal{L}, \eta) = f(\beta, \dot{\beta}, p, r, \delta_a, \delta_r) \quad (5.4)$$

The ensuing analysis is concerned with the question of lateral-directional static stability or the tendency of an airplane to return to stabilized flight after being perturbed in yaw or roll. This will be determined by the values of the yawing and rolling moments ( $\eta$  &  $\mathcal{L}$ ). Since the side force equation governs only the aircraft translatory response and has no effect on the angular motion, the side force equation will not be considered.

The two remaining aerodynamic functions can be expressed in terms of non-dimensional stability derivatives, angular rates and angular displacements:

$$C_n = C_{n\beta}\beta + C_{n\dot{\beta}}\dot{\beta} + C_{np}\hat{p} + C_{nr}\hat{r} + C_{n\delta_a}\delta_a + C_{n\delta_r}\delta_r \quad (5.5)$$

$$C_l = C_{l\beta}\beta + C_{l\dot{\beta}}\dot{\beta} + C_{lp}\hat{p} + C_{lr}\hat{r} + C_{l\delta_a}\delta_a + C_{l\delta_r}\delta_r \quad (5.6)$$

The analysis of aircraft lateral-directional motion is based on these two equations. A cursory examination of these equations reveals the presence of "cross-coupling" terms, e.g.,  $C_{np}$  and  $C_{n\delta_a}$  in the yawing moment equation (5.5). It is for this reason that aircraft lateral motions and directional motions must be considered together - each one influences the other.

Static directional stability will be considered first. Each stability derivative in equation (5.5) will be discussed and its contribution to aircraft stability will be analyzed. A summary of these stability derivatives is shown in figure 5.1.

DERIVATIVE	NAME	SIGN FOR A STABLE AIRCRAFT	CONTRIBUTING PARTS OF AIRCRAFT
$C_{n\beta}$	Static Directional Stability or Weathercock Stability	(+)	Tail, Fuselage, Wing
$C_{n\dot{\beta}}$	Lag Effects	(-)	Tail
$C_{np}$	Cross-Coupling	(+)	Wing, Tail
$C_{nr}$	Yaw Damping	(-)	Tail, Wing, Fuselage
$C_{n\delta_a}$	Adverse or Complimentary Yaw	"0" or slightly (+)	Lateral Control
$C_{n\delta_r}$	Rudder Power	(+)	Rudder Control

Figure 5.1

## 5.2 $C_{n\beta}$ - STATIC DIRECTIONAL STABILITY OR WEATHERCOCK STABILITY

Static directional stability is defined as the initial tendency of an aircraft to return to or depart from its equilibrium angle of sideslip when disturbed. Although the static directional stability of an aircraft is determined through consideration of all the terms in equation 5.5,  $C_{n\beta}$  is often referred to as "static directional stability" because it is the predominant term.

When an aircraft is placed in a sideslip, aerodynamic forces develop which create moments about all three axis. The moments created about the Z axis tend to turn the nose of the aircraft into or away from the relative wind. The aircraft is statically directionally stable if the moments created by a sideslip angle tend to align the nose of the aircraft with the relative wind. By convention, sideslip angle is defined as positive if the relative wind is displaced to the right of the fuselage reference line.

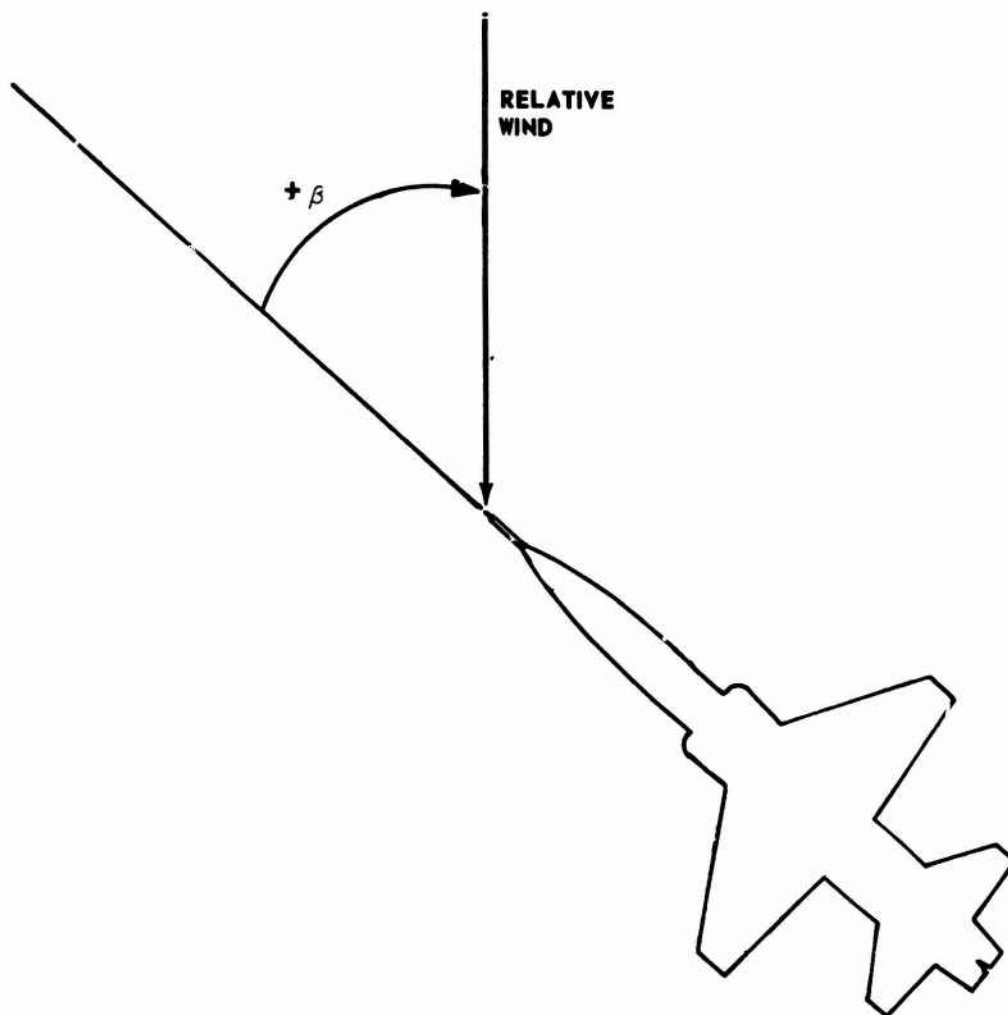


Figure 5.2

In figure 5.2 the aircraft is in a right sideslip. It is statically stable if it develops yawing moments that tend to align it with the relative wind, or, in this case, right (positive) yawing moments. Therefore, an aircraft is statically directionally stable if it develops positive yawing moments with a positive increase in sideslip. Thus, the slope of a plot of yawing moment coefficient,  $C_n$ , versus sideslip,  $\beta$ , is a quantitative measure of the static directional stability that an aircraft possesses. This plot would normally be determined from wind tunnel results.

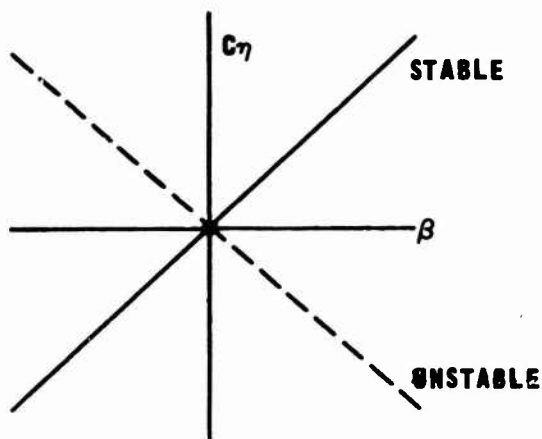


Figure 5.3  
WIND TUNNEL RESULTS OF YAWING  
MOMENT COEFFICIENT vs SIDESLIP

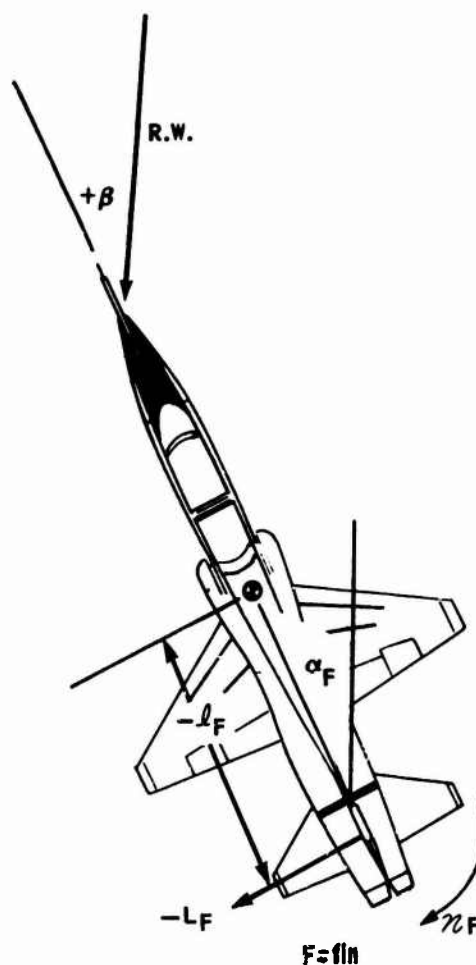


Figure 5.4

The total value of the directional stability derivative,  $C_{n\beta}$ , at any sideslip angle, is determined by contributions from the vertical tail, the fuselage, and the wing. These contributions will be discussed separately.

#### ● 5.2.1 VERTICAL TAIL CONTRIBUTION TO $C_{n\beta}$ :

The vertical tail is the primary source of directional stability for virtually all aircraft. When the aircraft is yawed, the angle of attack of the vertical tail is changed. This change in angle of attack produces a change in lift on the vertical tail, and thus a yawing moment about the Z-axis.

Referring to figure 5.4, the yawing moment produced by the tail is:

$$n_F = (-l_F) (-L_F) = l_F L_F \quad (5.7)$$

The minus signs in this equation arise from the use of the sign convention adopted in the study of aircraft equations of motion. Forces to the left and distances behind the aircraft cg are negative.

As in other aerodynamic considerations, it is convenient to consider yawing moments in coefficient form so that static directional stability can be evaluated independent of weight, altitude and speed. Putting equation 5.7 in coefficient form:

$$C_{n_F} = \frac{l_F L_F}{q_w S_w b_w} = \frac{l_F C_{L_F} q_F S_F}{q_w S_w b_w} \quad (5.8)$$

Vertical tail volume ratio,  $V_V$ , is defined as:

$$V_V = \frac{S_F l_F}{S_w b_w} \quad (5.9)$$

The sign of  $V_V$  may be either positive or negative. Making this substitution in equation 5.8:

$$C_{n_F} = \frac{C_{L_F} q_F V_V}{q_w} \quad (5.10)$$

For a propeller-driven aircraft,  $q_w$  is greater than  $q_F$ . However, for a jet aircraft, these two quantities are equal. Thus, for a jet aircraft, equation 5.10 becomes:

$$C_{n_F} = C_{L_F} V_V \quad (5.11)$$

The lift curve for a vertical tail is presented in figure 5.5.

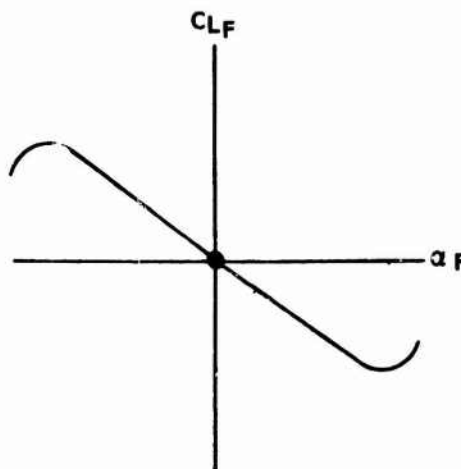


Figure 5.5 LIFT CURVE FOR A VERTICAL TAIL

The negative slope of this curve is a result of the sign convention used. Reference figure 5.4. When the relative wind is displaced to the right of the fuselage reference line, the vertical tail is placed at a positive angle of attack. However, this results in a lift force to the left, or a negative lift. Thus, the sign of the lift curve slope of a vertical tail,  $a_F$ , will always be negative below the stall.

$$C_{L_F} = a_F \alpha_F \quad (5.12)$$

Making this substitution in equation 5.11:

$$C_{n_F} = a_F \alpha_F V_v \quad (5.13)$$

The angle of attack of the vertical tail,  $\alpha_F$ , is not merely  $\beta$ . If the vertical tail were placed alone in an airstream, the  $\alpha_F$  would be equal to  $\beta$ . However, when the tail is installed on an aircraft, changes in both magnitude and direction of the local flow at the tail take place.

These changes may be caused by a propeller slipstream, or by the wing and the fuselage when the airplane is yawed. The angular deflection is allowed for by introducing the sidewash angle,  $\sigma$ , analogous to the downwash angle,  $\epsilon$ . The value of  $\sigma$  is very difficult to predict, therefore suitable wind tunnel tests are required. The sign of  $\sigma$  is defined as positive if it causes  $\alpha_F$  to be less than  $\beta$ . Thus,

$$\sigma = \beta - \alpha_F \quad (5.14)$$

Substituting in equation 5.13:

$$C_{n_F} = a_F V_v (\beta - \sigma) \quad (5.15)$$

The contribution of the vertical tail to weathercock stability is found by examining the change in  $C_{n_F}$  with a change in sideslip angle,  $\beta$ .

$$\frac{\partial C_{n_F}}{\partial \beta} = \left[ C_{n_\beta}(\text{Tail}) \right]_{\text{Fixed}} = V_v a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \quad (5.16)$$

The subscript "fixed" is added to emphasize that, thus far, the vertical tail has been considered as a surface with no movable parts, i.e., the rudder is "fixed."

Equation 5.16 reveals that the vertical tail contribution to directional stability can only be changed by varying the vertical tail volume ratio,  $V_v$ , or the vertical tail lift curve slope,  $a_F$ . The vertical tail volume ratio can be changed by varying the size of the vertical tail, or its distance from the aircraft cg. The vertical tail lift curve slope can be changed by altering the basic airfoil section of the vertical tail, or by end plating the vertical fin. An end plate on the top of the vertical tail is a relatively minor modification and yet it increases the directional stability of the aircraft significantly. This fact has been utilized in the case of the T-38 (figure 5.6). As can be seen in figure 5.7, the entire stabilator on the F-104 acts as an end plate and, therefore, adds greatly to the directional stability of the aircraft.

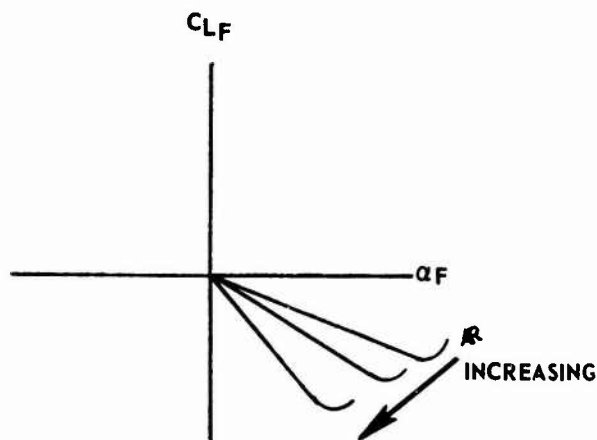


Figure 5.6



Figure 5.7

The effect of an end plate on the vertical stabilizer is to increase the effective aspect ratio of the vertical tail. As with any airfoil, this change in aspect ratio produces a change in the lift curve slope of the airfoil.



As the aspect ratio is increased, the  $\alpha_F$  for stall is decreased. Thus, if the aspect ratio of the vertical tail is too high, the vertical tail will stall at low sideslip angles and a large decrease in directional stability will occur.

### 5.2.2 FUSELAGE CONTRIBUTION TO $C_{n\beta}$ :

The subsonic center of pressure of a typical fuselage occurs about one-fourth of the distance back from the nose. Since the aircraft center of gravity usually lies behind this point, the fuselage is generally destabilizing.

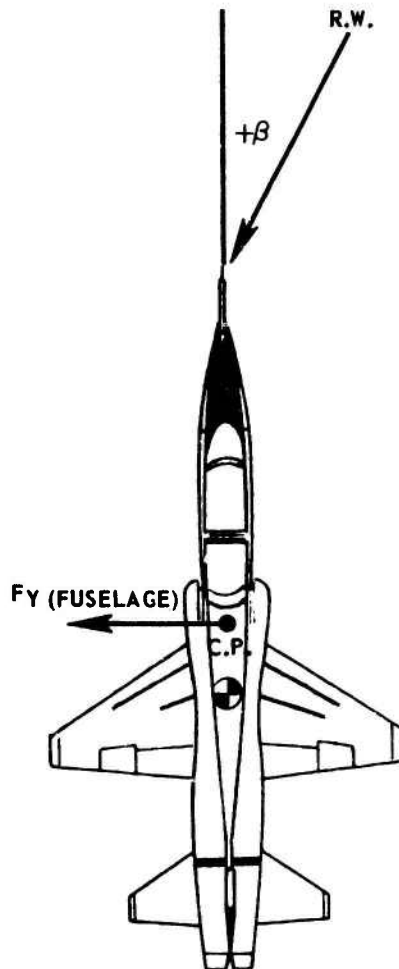


Figure 5.8

As can be seen from figure 5.8, a positive sideslip angle will produce a negative yawing moment about the cg, thus,  $C_{n\beta}$  (fuselage) is negative or destabilizing. The destabilizing influence of the fuselage diminishes at large sideslip angles due to a decrease in lift as the fuselage stall angle of attack is exceeded, and also due to an increase in parasite drag acting at the center of equivalent parasite area which is located aft of the cg.

If the overall directional stability of an aircraft becomes too low, the fuselage-tail combination can be made more stabilizing by adding a dorsal fin or a ventral fin. A dorsal fin was added to the C-123 and a ventral fin was added to the F-104 to improve static directional stability.

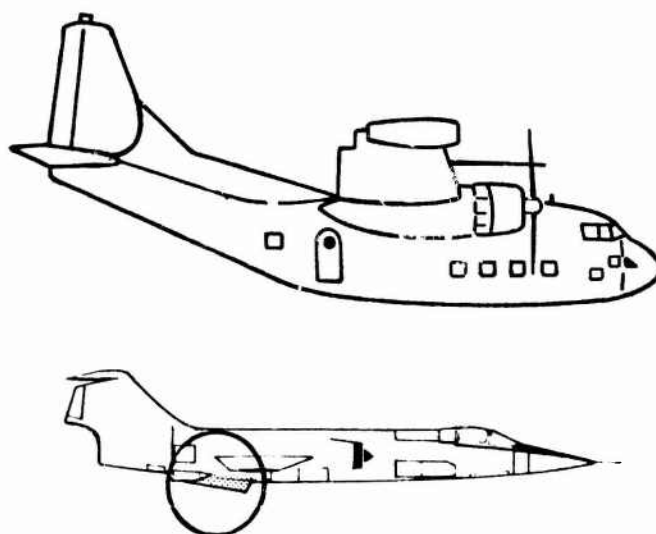


Figure 5.9

Since the addition of a dorsal fin decreases the effective aspect ratio of the tail, a higher sideslip angle can be attained before the vertical fin will stall. However, the major effect of the dorsal fin at large sideslip angles is to move the center of equivalent parasite area further aft of the cg, therefore producing a greater stabilizing moment at any given sideslip angle. Thus, a dorsal fin greatly increases directional stability at large sideslip angles. Figure 5.10 shows the effect on directional stability of adding a dorsal fin.

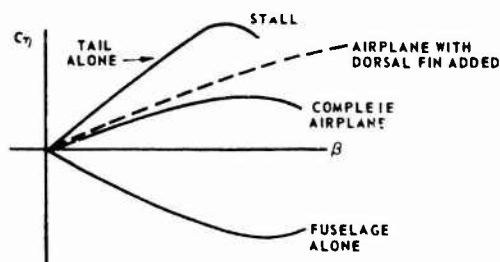


Figure 5.10 EFFECT OF ADDING A DORSAL FIN

$C_{n\beta}$  (fuselage) is difficult to estimate, and although some empirical formulas exist, it is usually measured directly by wind tunnel tests using a model without a tail.

### ● 5.2.3 WING CONTRIBUTION TO $C_{n\beta}$ :

The wing contribution to static directional stability is usually small. Straight wings make a slight positive contribution to static directional stability due to fuselage blanking in a sideslip. Effectively, the relative wind "sees" less of the downwind wing due to fuselage blanking. This reduces the lift of the downwind wing, and thus reduces the induced drag on the downwind wing. The difference in induced drag on the two wings tends to yaw the aircraft into the relative wind.

Swept back wings produce a greater positive contribution to static directional stability than do straight wings.

Reference figure 5.11. The wing sweep angle,  $\Lambda$  is defined as the angle between a perpendicular to the fuselage reference line and the quarter chord line of the wing.

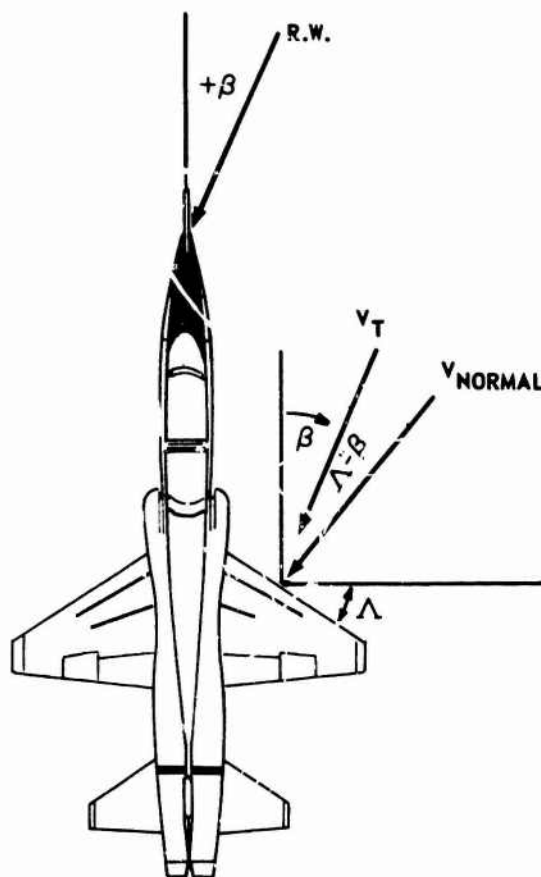


Figure 5.11

It can be seen that the component of free stream velocity normal to the wing is greater for swept back wings than for straight wings, and that is also greater on the upwind wing.

$$V_{N(\text{Upwind})} = V_T \cos (\Lambda - \beta) \quad (5.17)$$

$$V_{N(\text{Downwind})} = V_T \cos (\Lambda + \beta) \quad (5.18)$$

This difference in normal components creates a dissimilance of lift and therefore a disparity in induced drag on the two wings. Thus a stabilizing yawing moment is created. Similarly, forward swept wings would create an unstable contribution to static directional stability.

#### ● 5.2.4 MISCELLANEOUS EFFECTS ON $C_{n\beta}$ :

A propeller can have large effects on an aircraft's static directional stability. The propeller contribution to directional stability arises from the side force component at the propeller disc created as a result of yaw (figure 5.12)

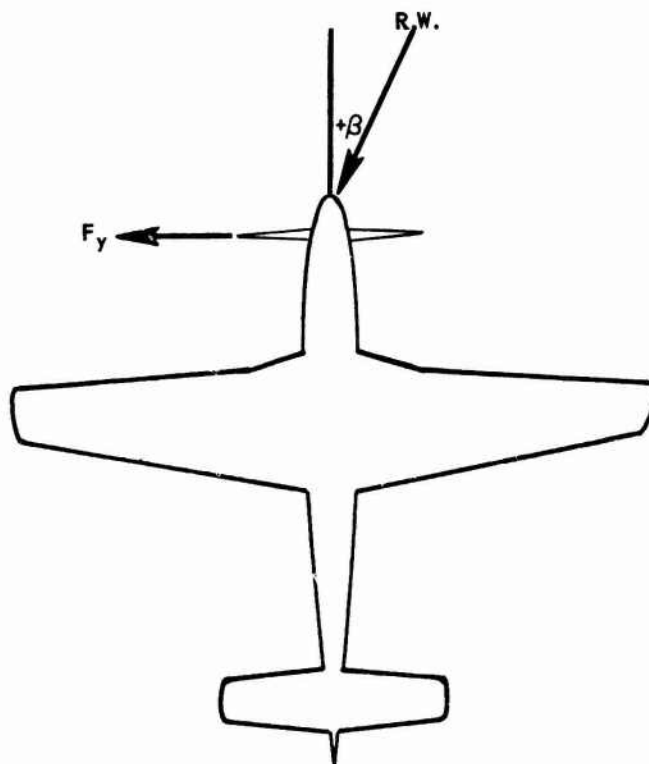


Figure 5.12

The propeller is destabilizing if a tractor and stabilizing if a pusher. Similarly, engine intakes have the same effects if they are located fore or aft of the aircraft cg.

Engine nacelles act like a small fuselage and can be stabilizing or destabilizing depending on whether their cp is located ahead or behind the cg.

Aircraft cg movement is restricted by longitudinal static stability considerations. However, within the relatively narrow limits established by longitudinal considerations, cg movements have no significant effects on static directional stability.

### ■ 5.3 $C_{n_Y}$ - RUDDER POWER

In most flight conditions, it is desired to maintain the sideslip angle equal to zero. If the aircraft has positive directional stability and is symmetrical, then it will tend to fly in this condition. However, yawing moments may act on the aircraft as a result of asymmetric thrust (one engine inoperative), slip stream rotation, or the unsymmetric flow field associated with turning flight. Under these conditions, sideslip angle can be kept to zero only by the application of a control moment. The control that provides this moment is the rudder.

Recall that,

$$C_{n_F} = a_F \alpha_F V_V \quad (5.13)$$

$$\frac{\partial C_{n_F}}{\partial \delta_r} = \frac{\partial C_n}{\partial \delta_r} = a_F V_V \frac{\partial \alpha_F}{\partial \delta_r} \quad (5.19)$$

Defining rudder effectiveness,  $T$ , as:

$$T = \frac{\partial \alpha_F}{\partial \delta_r} \quad (5.20)$$

$$\frac{\partial C_n}{\partial \delta_r} = C_{n_{\delta_r}} = a_F V_V T \quad (5.21)$$

The derivative,  $C_{n_{\delta_r}}$ , is called "rudder power" and by definition, its algebraic sign is always positive. This is because a positive rudder deflection,  $+\delta_r$  is defined as one that produces a positive moment about the cg,  $+C_n$ . The magnitude of the rudder power can be altered by varying the size of the vertical tail and its distance from the aircraft cg, or by using different airfoils for the tail and/or rudder, or by varying the size of the rudder.

### ■ 5.4 RUDDER FIXED STATIC DIRECTIONAL STABILITY

Having some knowledge of both  $C_{n_\beta}$  and  $C_{n_{\delta_r}}$ , it is now possible to work toward some relationship that can be used in flight to measure the

static directional stability of the aircraft. In flight, the maneuver that will be used to determine the static directional stability of the aircraft is the "steady straight sideslip." In a steady straight sideslip, equation 5.5 reduces to,

$$C_{n_{\beta}} \beta + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r = 0 \quad (5.22)$$

Thus,

$$\delta_r = - \frac{C_{n_{\beta}}}{C_{n_{\delta_r}}} \beta - \frac{C_{n_{\delta_a}}}{C_{n_{\delta_r}}} \delta_a \quad (5.23)$$

$$\frac{\partial \delta_r}{\partial \beta} = - \frac{C_{n_{\beta}(\text{Fixed})}}{C_{n_{\delta_r}}} \quad (5.24)$$

Again, the subscript "fixed" is added as a reminder that equation 5.24 is an expression for the static directional stability of an aircraft if the rudder is not free to float. Looking at equation 5.24,  $C_{n_{\delta_r}}$  is a known quantity once an aircraft is built, therefore,  $\partial \delta_r / \partial \beta$  can be taken as a direct indication of the rudder fixed static directional stability of an aircraft. The relationship,  $\partial \delta_r / \partial \beta$ , can easily be measured in flight. Since  $C_{n_{\beta}}$  has to be positive in order to have positive directional stability, and  $C_{n_{\delta_r}}$  is positive by definition,  $\partial \delta_r / \partial \beta$  must be negative to obtain positive directional stability.

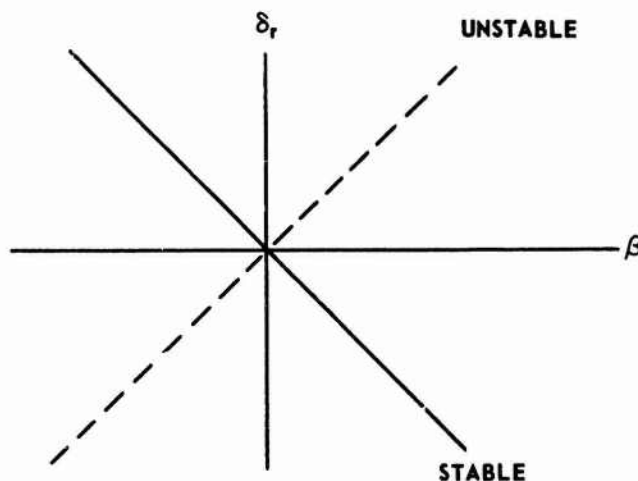


Figure 5.13 RUDDER DEFLECTION vs SLIDESLIP

## 6.5 RUDDER FREE DIRECTIONAL STABILITY

On aircraft with reversible control systems, the rudder is free to float in response to its hinge moments, and this floating can have large effects on the directional stability of the airplane. In fact, a plot of  $\partial \delta_r / \partial \beta$  may be stable while an examination of the rudder free static directional stability reveals the aircraft to be unstable. Thus, if the rudder is free to float, there will be a change in the tail contribution to static directional stability. To analyze the nature of this change, recall that hinge moments are produced by the pressure distribution caused by angle of attack and control surface deflection. In the case of the rudder,

$$H_m = H_{m_0} + \frac{\partial H_m}{\partial \alpha_F} \alpha_F + \frac{\partial H_m}{\partial \delta_r} \delta_r \quad (5.25)$$

In coefficient form

$$C_h = C_{h_{\alpha_F}} \cdot \alpha_F + C_{h_{\delta_r}} \cdot \delta_r \quad (5.26)$$

It can be seen that when the vertical tail is placed at some angle of attack,  $\alpha_F$ , the rudder will start to "float." However, as soon as it deflects, restoring moments are set up, and an equilibrium floating angle will be reached where the floating tendency is just balanced by the restoring tendency and  $C_h = 0$ . At this point,

$$C_{h_{\alpha_F}} \cdot \alpha_F = - C_{h_{\delta_r}} \cdot \delta_r(\text{Float}) \quad (5.27)$$

Thus,

$$\delta_r(\text{Float}) = - \frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}} \alpha_F \quad (5.28)$$

With this background, it is now possible to develop a relationship that expresses the static directional stability of an aircraft with the rudder free to float.

Recall that,

$$C_{n_F} = V_v a_F \alpha_F \quad (5.13)$$

$$\alpha_F = \beta - \sigma + \frac{\partial \alpha_F}{\partial \delta_r} \delta_r(\text{Float}) \quad (5.29)$$

Therefore,

$$C_{n_F} = V_v a_F \left( \beta - \sigma + \frac{\partial \alpha_F}{\partial \delta_r} \delta_r(\text{Float}) \right) \quad (5.30)$$

$$C_{n_{\beta}(\text{Free})} = \frac{\partial C_{n_F}}{\partial \beta} = V_V a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} + \tau \frac{\partial \delta r(\text{Float})}{\partial \beta} \right) \quad (5.31)$$

$$C_{n_{\beta}(\text{Free})} = V_V a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \cdot \left( 1 + \tau \frac{\partial \delta r(\text{Float})}{\partial \beta} \cdot \frac{1}{1 - \frac{\partial \sigma}{\partial \beta}} \right) \quad (5.32)$$

From equation 5.14,

$$\frac{\partial \alpha_F}{\partial \beta} = 1 - \frac{\partial \sigma}{\partial \beta} \quad (5.33)$$

$$C_{n_{\beta}(\text{Free})} = V_V a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \left( 1 + \tau \frac{\partial \delta r(\text{Float})}{\partial \beta} \cdot \frac{\partial \beta}{\partial \alpha_F} \right) \quad (5.34)$$

$$C_{n_{\beta}(\text{Free})} = V_V a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \cdot \left( 1 + \tau \frac{\partial \delta r(\text{Float})}{\partial \alpha_F} \right) \quad (5.35)$$

Recall that,

$$\delta r(\text{Float}) = - \frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}} \alpha_F \quad (5.28)$$

Therefore,

$$\frac{\partial \delta r(\text{Float})}{\partial \alpha_F} = - \frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}} \quad (5.36)$$

Thus,

$$C_{n_{\beta}(\text{Free})} = V_V a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \cdot \left( 1 - \tau \frac{C_{h_{\alpha_F}}}{C_{h_{\delta_r}}} \right) \quad (5.37)$$

It can be seen that this expression differs from equation 5.16, the expression for rudder fixed static directional stability by the term  $(1 - \tau C_{h_{\alpha_F}}/C_{h_{\delta_r}})$ . Since this term will always result in a quantity less than one, it can be stated that the effect of rudder float is to reduce the slope of the static directional stability curve.

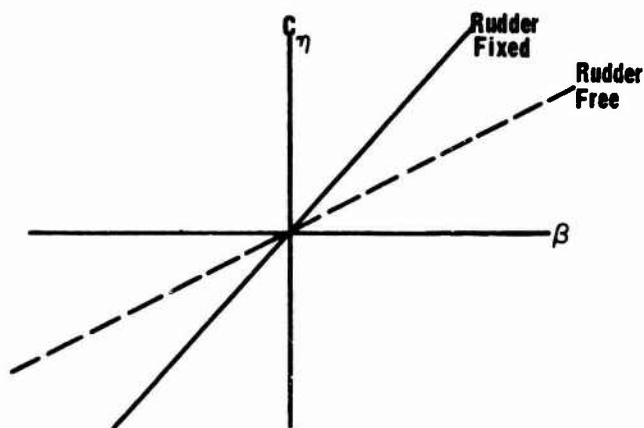


Figure 5.14

Equation 5.37 does not contain parameters that are easily measured in flight, therefore it is necessary to develop an expression that will be useful in flight test work.

Assuming a steady straight sideslip, figure 5.15 schematically represents the forces and moments at work.

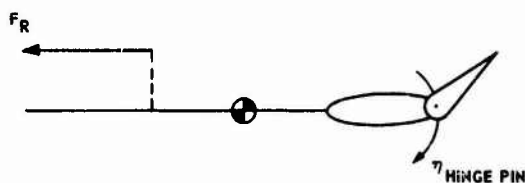


Figure 5.15

In a steady straight sideslip,  $\Sigma \eta = 0$ . Therefore, it follows that  $\Sigma \eta_{\text{Hinge Pin}} = 0$ . Now if moments are summed about the rudder hinge pin, the rudder force exerted by the pilot,  $F_R$ , acts through a moment arm and gearing mechanism, both accounted for by some constant,  $K$ , and must balance the other aerodynamic yawing moments so that  $\Sigma \eta_{\text{Hinge Pin}} = 0$ . The pilot is hindered in his task by the fact that the rudder floats. Thus, in steady straight flight,

$$\Sigma \eta_{\text{Hinge Pin}} = 0 = F_R \cdot K + H_m \quad (5.38)$$

$$F_R = -G \cdot H_M \quad (5.39)$$

Where  $G$  is merely  $1/K$ .

Knowing,

$$H_m = C_h q_r S_r c_r \quad (5.40)$$

From equation 5.26,

$$H_m = q_r S_r c_r (C_{h_{\alpha_F}} \cdot \alpha_F + C_{h_{\delta_r}} \cdot \delta_r) \quad (5.41)$$

Thus, equation 5.39 becomes,

$$\Gamma_r = - G q_r S_r c_r (C_{h_{\alpha_F}} \cdot \alpha_F + C_{h_{\delta_r}} \cdot \delta_r) \quad (5.42)$$

Applying equation 5.27,

$$F_r = - G q_r S_r c_r (-C_{h_{\delta_r}} \cdot \delta_{r(\text{Float})} + C_{h_{\delta_r}} \cdot \delta_r) \quad (5.43)$$

$$F_r = - G q_r S_r c_r C_{h_{\delta_r}} (\delta_r - \delta_{r(\text{Float})}) \quad (5.44)$$

The difference between where the pilot pushes the rudder,  $\delta_r$ , and the amount it floats,  $\delta_{r(\text{Float})}$ , is the free position of the rudder,  $\delta_{r(\text{Free})}$

Therefore,

$$F_r = - G q_r S_r c_r C_{h_{\delta_r}} \delta_{r(\text{Free})} \quad (5.45)$$

$$\frac{\partial F_r}{\partial \beta} = - G q_r S_r c_r C_{h_{\delta_r}} \cdot \frac{\partial \delta_{r(\text{Free})}}{\partial \beta} \quad (5.46)$$

From equation 5.24, it can be shown that,

$$\frac{\partial \delta_{r(\text{Free})}}{\partial \beta} = - \frac{C_{n_{\beta}}(\text{Free})}{C_{n_{\delta_r}}} \quad (5.47)$$

Thus,

$$\frac{\partial F_r}{\partial \beta} = G q_r S_r c_r \frac{C_{h_{\delta_r}}}{C_{n_{\delta_r}}} C_{n_{\beta}}(\text{Free}) \quad (5.48)$$

This equation shows that the parameter,  $\partial F_r / \partial \beta$ , can be taken as an indication of the rudder free static directional stability of an aircraft. This parameter can be readily measured in flight.

An analysis of the components of equation 5.48 reveals that for static directional stability, the sign of  $\partial F_r / \partial \beta$  should be negative.

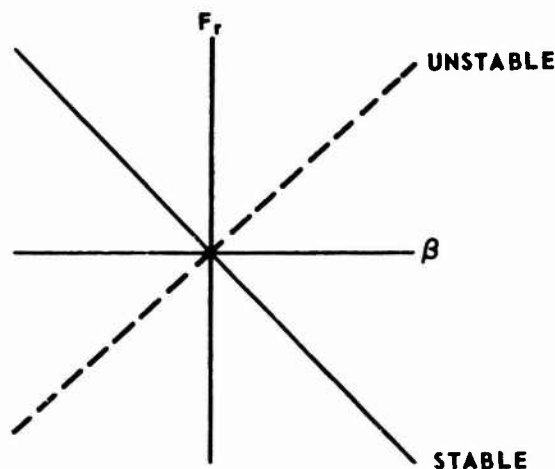


Figure 5.16

#### ■ 5.6 $C_{n\delta_a}$ - YAWING MOMENT DUE TO LATERAL CONTROL DEFLECTION

The remaining derivatives in equation 5.5 that have not been studied thus far are called "cross derivatives." It is the existence of these cross derivatives that causes the rolling and yawing motions to be so closely coupled.

The first of these cross derivatives to be covered will be  $C_{n\delta_a}$ , and is the yawing moment due to lateral control deflection. In order for a lateral control to produce a rolling moment, it must create an unbalanced lift condition on the wings. The wing with the most lift will also produce the most induced drag according to the equation  $C_{D_i} = C_L^2 / \pi e AR$ . Also, any change in the profile of the wing due to a lateral control deflection will cause a change in profile drag. Thus, any lateral control deflection will produce a change in both induced and profile drag. The predominate effect will be dependent on the particular aircraft configuration and the flight condition. If induced drag predominates, the aircraft will tend to yaw away from the direction of roll. This phenomenon is known as "adverse yaw." The sign of  $C_{n\delta_a}$  for adverse yaw is negative. If profile drag predominates, the aircraft will tend to yaw into the direction of roll. This is known as "complimentary" or "proverse" yaw. The sign of  $C_{n\delta_a}$  for complimentary yaw is positive. Both ailerons and spoilers are capable of producing either adverse or complimentary yaw. To determine which condition will prevail, the particular aircraft configuration and flight condition must be analyzed. If design permits, it is desirable to have  $C_{n\delta_a} = 0$  or be slightly positive. A slight positive value will ease the pilot's turn coordination task.

## 5.7 $C_{np}$ - YAWING MOMENT DUE TO ROLL RATE

The derivative  $C_{np}$  is called yawing moment due to roll rate. Both the wing and the tail contribute to  $C_{np}$ . The wing contribution arises from two sources. The first comes from the change in profile drag associated with the change in wing angle of attack due to rolling. As an aircraft is rolled, the angle of attack on the downgoing wing is increased. Refer to figure 5.17. Conversely, the angle of attack on the upgoing wing is decreased.

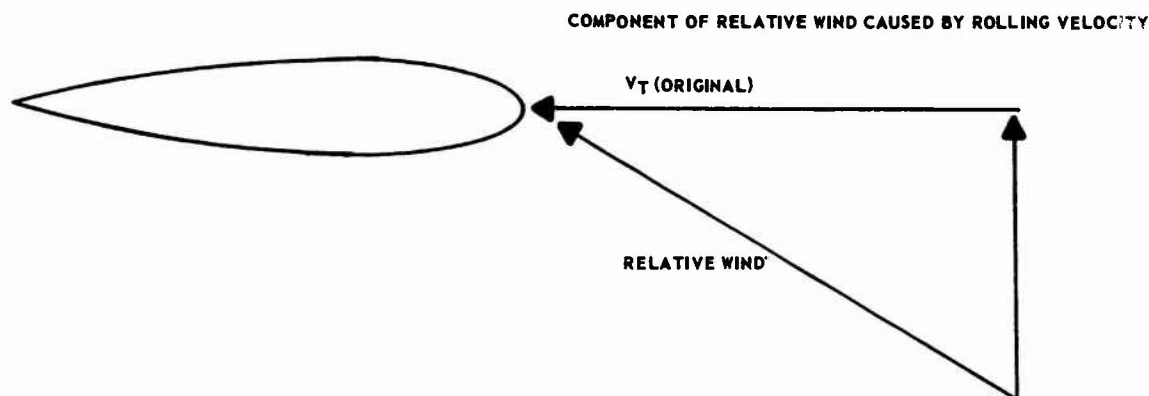
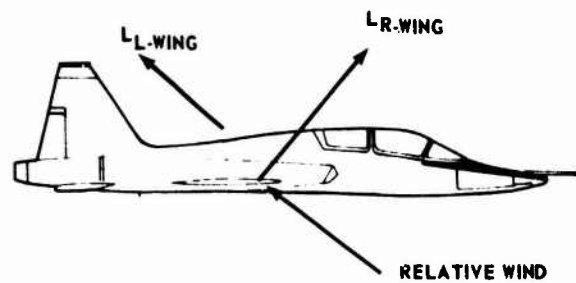


Figure 5.17

This increase in angle of attack on the downgoing wing means that the relative wind "sees" more of the downgoing wing and that therefore the profile drag will be greater on this wing than on the upgoing wing. For the right roll depicted in figure 5.17, the increased profile drag would cause a yaw to the right. Thus, the sign of  $C_{np}$  due to this effect only is positive. However, the second wing effect is predominant and the foregoing effect exerts only a mitigating influence.

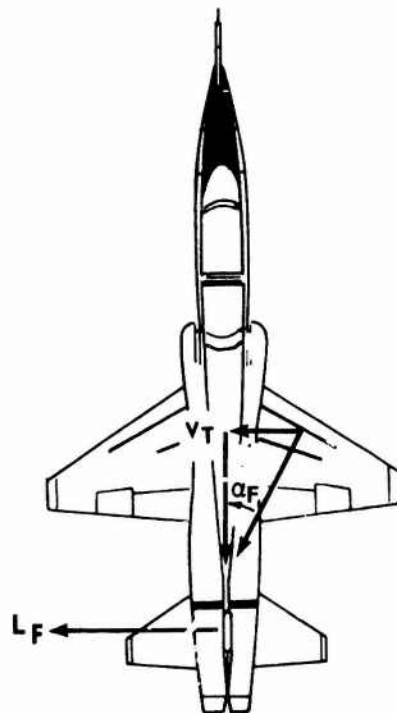
The local lift vector is always perpendicular to the local relative wind. As already discussed, the inclination of the relative wind is different on the wings during a roll. Thus, there will be a difference in the inclination of the two wing lift vectors. The lift vector on the downgoing wing will be tilted forward, and the lift vector on the upgoing wing will be tilted aft. Refer to figure 5.18.

Since each lift vector has a component in the X-direction, a yawing moment will result. In the case depicted, for a right roll the yaw will be to the left. Thus, the sign of  $C_{np}$  due to this effect will be negative. As previously mentioned, this is the predominate wing effect and thus, overall, the sign of the wing contribution to  $C_{np}$  is negative.



**Figure 5.18 INCLINATION OF WING LIFT VECTORS DURING A RIGHT ROLL**

The vertical tail makes a larger contribution to  $C_{np}$  than does either wing effect. Rolling changes the angle of attack on the vertical tail. Refer to figure 5.19.



**Figure 5.19 CHANGE IN ANGLE OF ATTACK OF THE VERTICAL TAIL DUE TO A RIGHT ROLL RATE**

This change in angle of attack on the vertical tail will generate a lift force. In the situation depicted in figure 5.19, the change in angle of attack will generate a lift force,  $L_F$ , to the left. This will create a positive yawing moment. Thus,  $C_{n_p}$  for the vertical tail is positive.

Considering both wing and tail, a slight positive value of  $C_{n_p}$  is desired to aid in Dutch roll damping.

#### ■ 5.8 $C_{n_r}$ YAW DAMPING

The derivative  $C_{n_r}$ , is called yaw damping and, by definition, its sign is always negative. The aircraft fuselage adds a negligible amount to  $C_{n_r}$  except when it is very large. The important contributions are those of the wing and tail.

The tail contribution to  $C_{n_r}$  arises from the fact that there is a change in angle of attack on the vertical tail whenever the aircraft is yawed. This change in  $\alpha_F$  produces a lift force,  $L_F$ , that in turn produces a yawing moment that opposes the original yawing moment. Refer to figure 5.20. The tail contribution to  $C_{n_r}$  accounts for 80-90% of the total "yaw damping" on most aircraft.

The wing contribution to  $C_{n_r}$  arises from the fact that in a yaw, the outside wing experiences an increase in both induced drag and profile drag due to the increased dynamic pressure on the wing. An increase in drag on the outside wing creates a yawing moment that opposes the original direction of yaw.

#### ■ 5.9 $C_{n_{\dot{\beta}}}$ - YAW DAMPING DUE TO LAG EFFECTS IN SIDEWASH

The derivative  $C_{n_{\dot{\beta}}}$  is yaw damping due to lag effects in sidewash,  $\sigma$ . Very little can be authoritatively stated about the magnitude or algebraic sign of  $C_{n_{\dot{\beta}}}$  due to the wide variations of opinion in interpreting the experimental data concerning it.

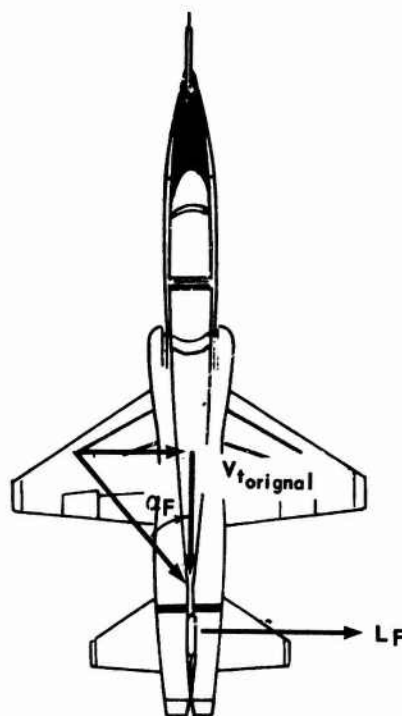


Figure 5.20  
CHANGE IN ANGLE OF ATTACK OF  
VERTICAL FIN DUE TO YAWING RATE

During any change in  $\beta$ , the angle of attack of the vertical fin will always be less than it will be at steady state. This is due to lag effects in sidewash. Since this phenomenon reduces the angle of attack of the vertical tail, it also reduces the yawing moment created by the vertical tail. This reduction in yawing moment is, effectively, a contribution to yaw damping. Thus the description, "yaw damping due to lag effects in sidewash."

#### ■ 5.10 HIGH SPEED ASPECTS OF STATIC DIRECTIONAL STABILITY

$C_{n\dot{\beta}}$  - The effectiveness of an airfoil decreases as the velocity increases supersonically. Thus, for a given  $\beta$ , as Mach increases, the restoring moment generated by the tail diminishes. The wing-fuselage combination continues to be destabilizing throughout the flight envelope. Thus, the overall  $C_{n\beta}$  of the aircraft will decrease with increasing Mach.

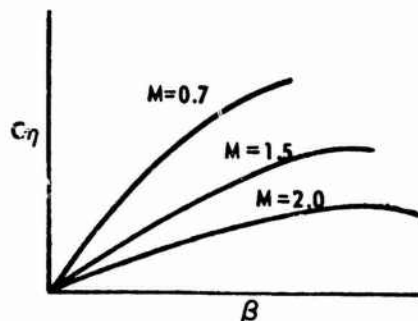


Figure 5.21 CHANGE IN  $C_{\eta\beta}$  WITH MACH NUMBER

The requirement for large values of  $C_{n\beta}$  is compounded by the tendency of high speed aerodynamic designs toward divergencies in yaw due to roll coupling effects. This problem can be combated by designing an extremely large tail (F-104, F-111, T-38), by endplating the tail (F-104, T-38), by using ventral fins (F-104), or by using fore body strakes.

The F-104 employs a ventral fin in addition to a sizeable vertical stabilizer to increase supersonic directional stability. The efficiency of underbody surfaces is not affected by wing wake at high angles of attack, and supersonically, they are located in a high energy compression pattern.

Fore body strakes located radially along the horizontal center line in the x-y plane of the aircraft have also been employed effectively to increase directional stability at supersonic speeds. This increase in  $C_{n\beta}$  by the employment of strakes is a result of a more favorable pressure distribution over the fore body surface, and in addition, the creation of improved flow effects at the vertical tail location by virtue of diminished flow circulation. In addition, even small sideslip angles will produce fuselage blanking of the downwind strake and create a dissimilarity of induced drag, and thus a stable contribution to  $C_{n\beta}$ .

$C_{n\delta_r}$  - In the transonic region, flow separation will decrease the effectiveness of any trailing edge control surface. On most aircraft however, this is offset by an increase in the  $C_{L_\alpha}$  curve in the transonic region. As a result, flight controls are usually the most effective in this region. However, as Mach number continues to increase, the  $C_{L_\alpha}$  curve will decrease, and thus, control surface effectiveness will continue to decrease. In addition, once the flow over the surface is supersonic, a trailing edge control cannot influence the pressure distribution on the surface itself, due to the fact that pressure disturbances cannot be transmitted forward in a supersonic environment. Thus, the rudder power will decrease as Mach increases above the transonic region.

$C_{n\delta_a}$  - For the same reasons discussed under rudder power, a given aileron deflection will not produce as much lift at high Mach number as it did transonically. Therefore, induced drag will be less. In addition, the profile drag, for a given aileron deflection, increases with Mach number. Thus, the tendency toward complimentary yaw increases with Mach.

$C_{n_r}$  - The development of yaw damping depends on the ability of the wing and tail to develop lift. Thus, as Mach number increases and the ability of all surfaces to develop lift decreases, yaw damping will also decrease.

$C_{n_p}$  - The slope of a curve of  $C_{n_p}$  normally doesn't change with Mach number. However, the magnitude of attainable roll rate will decrease with decreasing aileron effectiveness. Therefore, the magnitude of  $C_{n_p}$  encountered at higher Mach numbers will normally be less.

$C_{n_\beta}$  - This derivative normally will not change with Mach number.

## 5.11 STATIC LATERAL STABILITY

The analysis of aircraft lateral static stability is based on equation 5.6, which is repeated here for reference.

$$C_\ell = C_{\ell_\beta} \beta + C_{\ell_{\dot{\beta}}} \dot{\beta} + C_{\ell_p} \hat{p} + C_{\ell_r} \hat{r} + C_{\ell_{\delta_a}} \delta_a + C_{\ell_{\delta_r}} \delta_r \quad (5.6)$$

It can be seen that the rolling moment,  $C_\ell$ , is not a function of bank angle,  $\phi$ . In other words, a change in bank angle will produce no change in rolling moment. In fact,  $\phi$  produces no moment at all. Thus,  $C_{\ell_\phi} = 0$ , and although it is analogous to  $C_{n_\alpha}$  and  $C_{n_\beta}$ , it contributes nothing to the lateral static stability analysis.

Bank angle,  $\phi$ , does have an indirect effect on rolling moment. As the aircraft is rolled into a bank angle, a component of aircraft weight will act along the Y-axis, and will thus produce an unbalanced force. Refer to figure 5.22. This unbalanced force in the Y direction,  $F_Y$ , will produce a sideslip,  $\beta$ , and as seen from equation 5.6, this will influence the rolling moment produced.

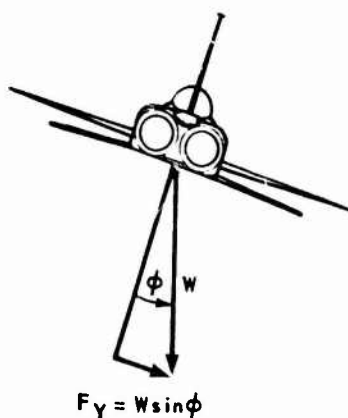


Figure 5.22 SIDE FORCE PRODUCED BY BANK ANGLE

Each stability derivative in equation 5.6 will be discussed and its contribution to aircraft stability will be analyzed. A summary of these stability derivatives follows:

DERIVATIVE	NAME	SIGN FOR A STABLE AIRCRAFT	CONTRIBUTING PARTS OF AIRCRAFT
$C_{l\beta}$	Dihedral Effect	(-)	Wing, Tail
$C_{l\dot{\beta}}$	$C_l$ due to $\dot{\beta}$	(+)	Wing, Tail
$C_{l_p}$	Roll Damping	(-)	Wing, Tail
$C_{l_r}$	$C_l$ due to Yaw Rate	(+)	Wing, Tail
$C_{l\delta_a}$	Lateral Control Power	(+)	Lateral Control
$C_{l\delta_r}$	$C_l$ due to Rudder Deflection	(-)	Rudder

Figure 5.23

#### 5.12 $C_{l\beta}$ - DIHEDRAL EFFECT

The tendency of an aircraft to fly wings level is related to the derivative  $C_{l\beta}$ , which is known as "Dihedral Effect." Although the static lateral stability of an aircraft is a function of all the derivatives in equation 5.6,  $C_{l\beta}$  is the predominant term. Therefore, static lateral stability is often referred to as "Stable Dihedral Effect."

An aircraft has stable dihedral effect if a positive sideslip produces a negative rolling moment. Thus, the algebraic sign of  $C_{l\beta}$  must be negative for stable dihedral effect.

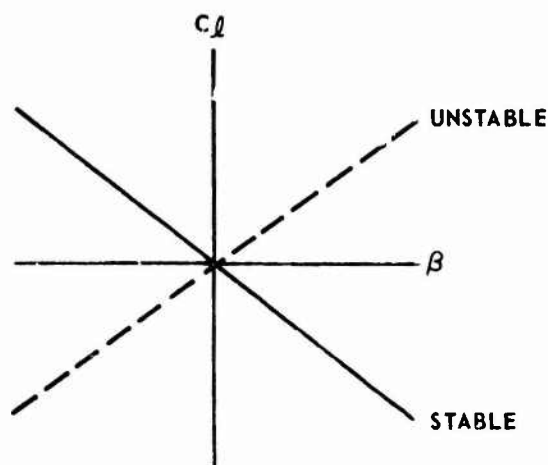


Figure 5.24 WIND TUNNEL RESULTS OF ROLLING MOMENT COEFFICIENT vs SIDESLIP

It is possible to have too much or too little dihedral effect. High values of dihedral effect give good spiral stability. If an aircraft has a large amount of positive dihedral effect, the pilot is able to pick up a wing with top rudder. This also means that in level flight a small amount of sideslip will cause the aircraft to roll and this can be annoying to the pilot. This is known as a high  $\phi/\beta$  ratio. In multi-engine aircraft, an engine failure will normally produce a large sideslip angle. If the aircraft has a great deal of dihedral effect, the pilot must supply an excessive amount of aileron force and deflection to overcome the rolling moment due to sideslip. Still another detrimental effect of too much dihedral effect may be encountered when the pilot rolls an aircraft. If an aircraft in rolling to the right tends to yaw to the left, the resulting right sideslip, together with stable dihedral effect, creates a rolling moment to the left. This effect could materially reduce the maximum roll rate available. The pilot, then wants a certain amount of dihedral effect, but not too much. The end result is usually a design compromise.

Both the wing and the tail exert an influence on  $C_{l\beta}$ . The various effects on  $C_{l\beta}$  can be classified as "direct" or "indirect." A direct effect actually produces some increment of  $C_{l\beta}$  while an indirect effect merely alters the value of the existing  $C_{l\beta}$ .

The discrete wing and tail effects that will be considered are classified as follows:

#### Effects on $C_{l\beta}$

<u>DIRECT</u>	<u>INDIRECT</u>
Geometric Dihedral	Aspect Ratio
Wing Sweep	Taper Ratio
Wing-Fuselage Interference	Tip Tanks
Vertical Tail	Wing Flaps

Figure 5.25

Geometric dihedral,  $\gamma$ , is defined as positive when the chord lines of the wing tip are above those at the wing root. To understand the effect of geometric dihedral on static lateral stability, consider figure 5.26.

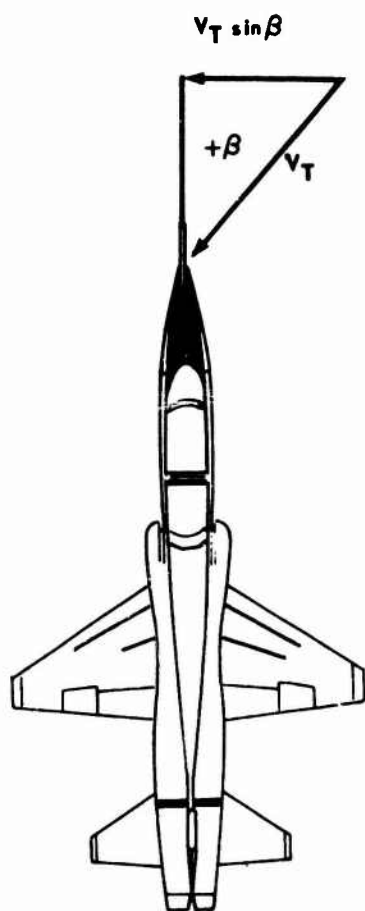


Figure 5.26a

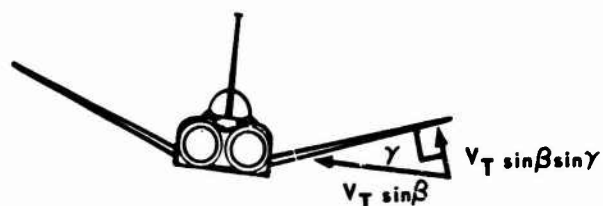


Figure 5.26b

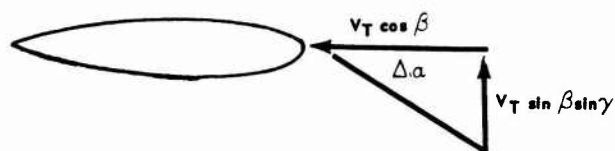


Figure 5.26c

It can be seen that when an aircraft is planed in a sideslip, positive geometric dihedral causes the component,  $V_T \sin \beta \sin \gamma$  to be added to the lift producing component of the relative wind,  $V_T \cos \beta$ . Thus, geometric dihedral causes the angle of attack on the upwind wing to be increased by  $\Delta\alpha$ .

$$\tan \Delta\alpha = \frac{V_T \sin \beta \sin \gamma}{V_T \cos \beta} = \tan \beta \sin \gamma \quad (5.49)$$

Making the small angle assumption,

$$\Delta\alpha = \tan \beta \sin \gamma \quad (5.50)$$

Conversely, the angle of attack on the downwind wing will be reduced. These changes in angle of attack tend to increase the lift on the upwind wing and decrease the lift on the downwind wing, thus producing a roll away from the sideslip. In figure 5.26, for example, a positive sideslip,  $+\beta$ , will increase the angle of attack on the upwind, or right, wing, thus producing a roll to the left. Therefore, it can be seen that this effect produces a stable, or negative, contribution to  $C_{l\beta}$ .

#### ● 5.12.1 WING SWEEP:

The wing sweep angle,  $\Lambda$ , is measured from a perpendicular to the aircraft x-axis at the forward wing root, to a line connecting the quarter cord points of the wing. Wing sweep back is defined as positive.

Aerodynamic theory shows that the lift of a yawed wing is determined by the component of the free stream velocity normal to wing. That is,  $L = C_L \frac{1}{2} \rho V_N^2 S$  where,  $V_N$  is the normal velocity.

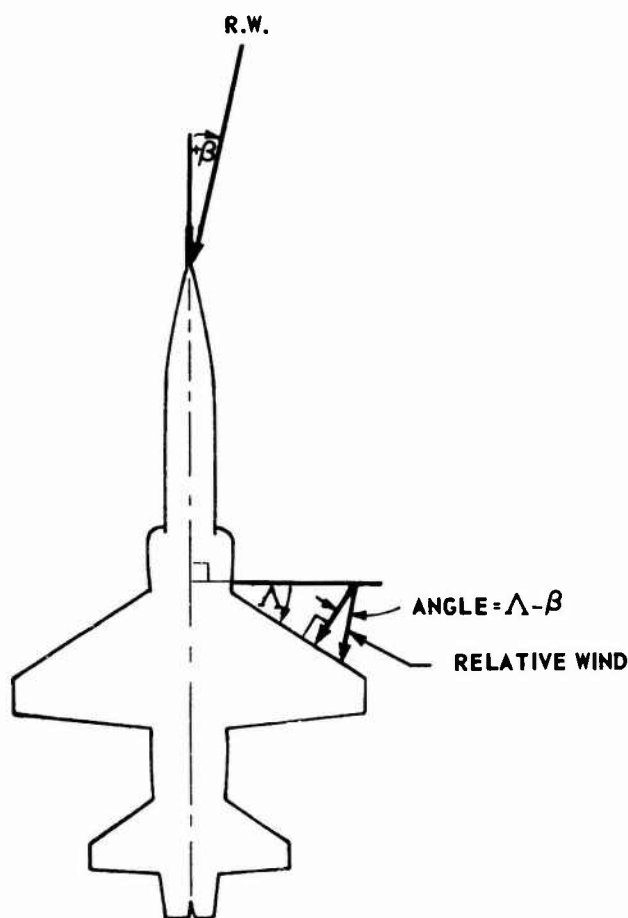


Figure 5.27 EFFECT OF WING SWEEP ON  $C_{l\beta}$

It can be seen from figure 5,27 that on a swept wing aircraft, the normal component of free stream velocity on the upwind wing is,

$$V_N = V_T \cos (\Lambda - \beta) \quad (5.51)$$

Conversely, on the downwind wing,

$$V_N = V_T \cos (\Lambda + \beta) \quad (5.52)$$

Therefore,  $V_N$  will be greater on the upwind wing. This will cause the upwind wing to produce more lift and will thus create a roll away from the direction of the sideslip. In other words, a right sideslip will produce a roll to the left. Thus, wing sweep makes a stable contribution to  $C_{l\beta}$  and produces the same effect as geometric dihedral.

To fully appreciate the effect of wing sweep on static lateral stability, it will be necessary to develop an equation relating the two.

$$L_{(\text{Upwind Wing})} = C_L \frac{S}{2} \frac{1}{2} \rho V_N^2 \quad (5.53)$$

$$L_{(\text{Upwind Wing})} = C_L \frac{S}{2} \frac{1}{2} \rho \left[ V_T \cos (\Lambda - \beta) \right]^2 \quad (5.54)$$

$$\Delta L = C_L \frac{S}{2} \frac{1}{2} \rho \left[ V_T \cos (\Lambda - \beta) \right]^2 - C_L \frac{S}{2} \frac{1}{2} \rho \left[ V_T \cos (\Lambda + \beta) \right]^2 \quad (5.55)$$

$$\Delta L = C_L \frac{S}{2} \frac{1}{2} \rho V_T^2 \left[ \cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta) \right] \quad (5.56)$$

Applying a trigonometric identity,

$$\left[ \cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta) \right] = \sin 2 \Lambda \sin 2 \beta \quad (5.57)$$

Making the assumption of a small sideslip angle,

$$\left[ \cos^2 (\Lambda - \beta) - \cos^2 (\Lambda + \beta) \right] = 2 \beta \sin 2 \Lambda \quad (5.58)$$

Therefore, equation 5.56 becomes,

$$\Delta L = C_L \frac{S}{2} \frac{1}{2} \rho V_T^2 2 \beta \sin 2 \Lambda = C_L \frac{S}{2} \frac{1}{2} \rho V_T^2 \beta \sin 2 \Lambda \quad (5.59)$$

The rolling moment produced by this change in lift is,

$$\mathcal{L} = - \Lambda L \cdot Y \quad (5.60)$$

Where  $\bar{Y}$  is the distance from the wing cp to the aircraft cg. The minus sign arises from the fact that equation 5.59 assumes a positive sideslip,  $+\beta$ , and for an aircraft with stable dihedral effect, this will produce a negative rolling moment.

$$C_{\ell} = \frac{\mathcal{L}}{q_w S_w b_w} \quad (5.61)$$

$$C_{\ell} = \frac{\bar{Y} C_L S \rho V_T^2 \beta \sin \Lambda}{\rho V_T^2 S b} = - \frac{C_L \bar{Y} \beta}{b} \sin \Lambda \quad (5.62)$$

$$\frac{\partial C_{\ell}}{\partial \beta} = C_{\ell \beta} = - \frac{\bar{Y}}{b} C_L \sin 2 \Lambda = - \text{CONST} (C_L \sin \Lambda) \quad (5.63)$$

Where the constant will be on the order of 0.2. Equation 5.63 should not be used above  $\Lambda = 45^\circ$  because highly swept wings are subject to leading edge separation at high angles of attack, and this can result in reversal of the dihedral effect. Therefore, it's best to use empirical results above  $\Lambda = 45^\circ$ .

From equation 5.63, it can be seen that at low speeds, high  $C_L$ , sweepback makes a large contribution to stable dihedral effect. However, at high speeds, low  $C_L$ , sweepback makes a relatively small contribution to stable dihedral effect.

For angles of sweep on the order of  $45^\circ$ , the wing sweep contribution to  $C_{\ell \beta}$  may be on the order of  $-1/5 C_L$ . For large values of  $C_L$ , this is a very large contribution, equivalent to nearly ten degrees of geometric dihedral. At very high angles of attack, such as during landing and takeoff, this effect can be very helpful to a swept wing fighter encountering downwash.

Since the effect of sweepback varies with  $C_L$ , becoming extremely small at high speeds, it can help keep the proper ratio of  $C_{\ell \beta}$  to  $C_{n \beta}$  at high speeds and reduce poor Dutch roll characteristics at these speeds.

#### 5.12.2 WING ASPECT RATIO:

The wing aspect ratio exerts an indirect effect on dihedral effect. On a high aspect ratio wing, the center of pressure is further from the cg than on a low aspect ratio wing. This results in high aspect ratio planforms having a longer moment arm and thus, greater rolling moments for a given asymmetric lift distribution. Refer to figure 5.28. It should be noted that aspect ratio, in itself, does not create dihedral effect, but that it merely alters the magnitude of the existing dihedral effect.

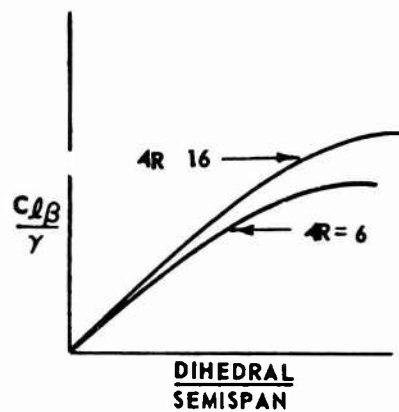


Figure 5.28

CONTRIBUTION OF ASPECT  
RATIO TO DIHEDRAL EFFECT.

### ● 5.12.3 WING TAPER RATIO:

Taper ratio,  $\lambda$ , is a measure of how fast the wing chord shortens. Taper ratio is the ratio of the tip chord to the root chord. Therefore, the lower the taper ratio, the faster the chord shortens. On highly tapered wings, the center of pressure is closer to the aircraft cg than on untapered wings. This results in a shorter moment arm and thus, less rolling moment for a given asymmetric lift distribution. Refer to figure 5.29. Taper ratio does not create dihedral effect, but merely alters the magnitude of the existing dihedral effect. Thus it has an "indirect" effect on dihedral effect.

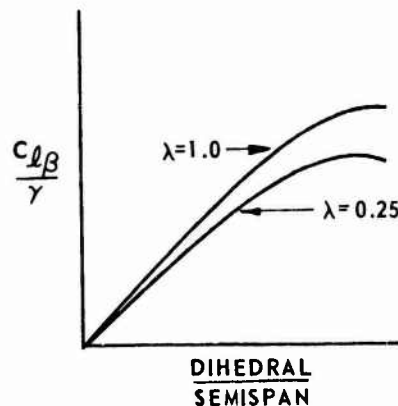


Figure 5.29 CONTRIBUTION OF TAPER RATIO TO DIHEDRAL EFFECT

#### ● 5.12.4 TIP TANKS

Tip tanks, pylon tanks and other external stores will generally exert an indirect influence on  $C_{l\beta}$ . Adding external stores creates an end-plate effect on the wing, and this, in turn, alters the effective aspect ratio of the wing. The effect of a given external store configuration is hard to predict analytically, and it is usually necessary to rely on empirical results. To illustrate the effect of various external store configurations, data for the F-80 is presented in figure 5.30. The data is for a clean F-80 230 gallon centerline tip tanks, and 165 gallon underslung tanks. This data shows that the centerline tanks increase dihedral effect while the underslung tanks reduce stable dihedral effect considerably.

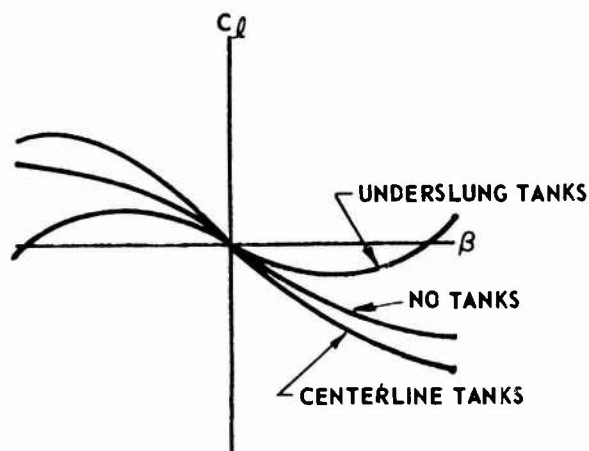


Figure 5.30 EFFECT OF TIP TANKS ON  $C_{l\beta}$  OF F-80

#### ● 5.12.5 PARTIAL SPAN FLAPS:

Partial-span flaps indirectly exert a detrimental effect on static lateral stability. Refer to figure 5.31.

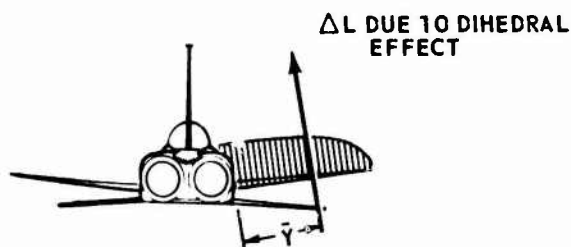


Figure 5.31a WING LIFT DISTRIBUTION, NO FLAPS

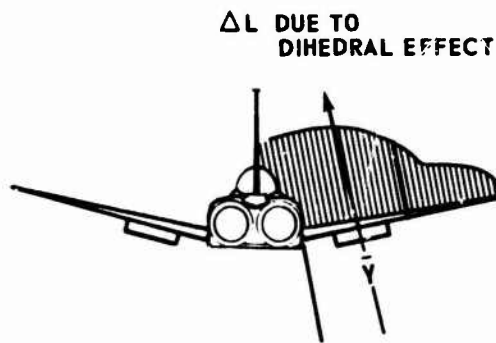


Figure 5.31b WING LIFT DISTRIBUTION, FLAPS EXTENDED

Partial-span flaps shift the center of lift of the wing inboard, reducing the effective moment arm  $\bar{Y}$ . Therefore, although the values of  $\Delta L$  remain the same, the rolling moment will decrease. The higher the effectiveness of the flaps in increasing the lift coefficient, the greater will be the change in span lift distribution and the more detrimental will be the effect of the flaps. Therefore, the decrease in lateral stability due to flap deflection may be large.

Deflected flaps cause a secondary variation in the effective dihedral that depends on the planform of the flaps themselves. If the shape of the wing gives a sweepback to the leading edge of the flaps, a slight positive dihedral effect results when the flaps are deflected. If the leading edge of the flaps are swept forward, flap deflection causes a slight negative dihedral effect. These effects are produced by the same phenomenon that produced a change in  $C_{l\beta}$  with wing sweep. The effect of flap platform on  $C_{l\beta}$  is generally small.

#### 5.12.6 WING - FUSELAGE INTERFERENCE :

Of the various interference effects between parts of the aircraft, probably the most important is the change in angle of attack of the wing near the root due to the flow pattern about the fuselage in a sideslip. To visualize the change in angle of attack, refer to figure 5.32.

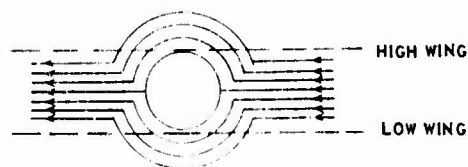


Figure 5.32 INFLUENCE OF WING - FUSELAGE INTERFERENCE ON  $C_{l\beta}$

The fuselage induces vertical velocities in a sideslip which, when combined with the mainstream velocity, alter the local angle of attack of the wing. When the wing is located at the top of the fuselage (high-wing), then the angle of attack will be increased at the wing root, and a positive sideslip will produce a negative rolling moment: i.e., the dihedral effect will be enhanced. Conversely, when the aircraft has a low wing, the dihedral effect will be diminished by the fuselage interference. Generally, this explains why high-wing airplanes often have little or no geometric dihedral, whereas low-wing aircraft may have a great deal of geometric dihedral.

#### ● 5.12.7 VERTICAL TAIL:

When an aircraft sideslips, the angle of attack of the vertical tail is changed. This change in angle of attack produces a lift force on the vertical tail. If the center of pressure of the vertical tail is above the aircraft cg, this lift force will produce a rolling moment. Refer to figure 5.33.

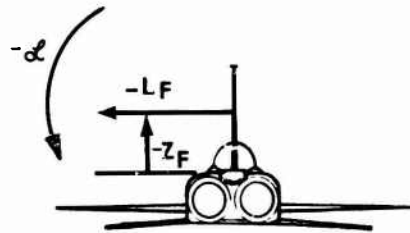


Figure 5.33 ROLLING MOMENT CREATED BY VERTICAL TAIL AT A POSITIVE ANGLE OF SIDESLIP

In the situation depicted in figure 5.33, the negative rolling moment was created by a positive sideslip angle, thus, the vertical tail contributes a stable increment to dihedral effect. This contribution can be quite large. In fact, it can be the major contribution to  $C_{l_\beta}$  on aircraft with large vertical tails such as the F-104 and the T-38. This effect can be calculated in the same manner yawing moments were calculated in the directional case.

Assuming a positive sideslip angle,

$$-\mathcal{L}_F = (-L_F) (-z_F) \quad (5.64)$$

$$C_{l_F} = \frac{-z_F L_F}{q_w S_w b_w} \quad (5.65)$$

$$C_{l_F} = \frac{-z_F C_{L_F} q_F S_F}{q_w S_w b_w} \quad (5.66)$$

Define  $V_F$  as,

$$V_F = \frac{S_F z_F}{S_W b_W} \quad (5.67)$$

Assume that for a jet aircraft,

$$q_F = q_W \quad (5.68)$$

And equation 5.66 becomes,

$$C_{l_F} = - C_{L_F} V_F = - a_F \alpha_F V_F \quad (5.69)$$

Knowing

$$\alpha_F = (\beta - \sigma) \quad (5.14)$$

$$C_{l_F} = - a_F V_F (\beta - \sigma) \quad (5.70)$$

$$C_{l_{\beta \text{ Vertical tail}}} = \frac{\partial C_{l_F}}{\partial \beta} = - a_F V_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \quad (5.71)$$

Equation 5.71 reveals that a vertical tail contributes a stable increment to  $C_{l_{\beta}}$ , whereas a ventral fin [ $V_F = (+)$ ] would contribute an unstable increment to  $C_{l_{\beta}}$ . Also, if the lift curve slope of the vertical tail is increased, by end plating for example, the stable dihedral effect would be greatly increased. For example, the F-104 has a high horizontal stabilizer that acts as an end plate on the vertical tail and this increases the stable dihedral effect. In fact, the increase is so large that it is necessary to add negative geometric dihedral to the wings and a ventral fin to maintain a reasonable value of stable dihedral effect.

### ■ 5.13 $C_{l_{\delta_a}}$ - LATERAL CONTROL POWER

Lateral control is achieved by altering the lift distribution so that the total lift on the two wings differ, thereby creating a rolling moment. This may be done simply by destroying a certain amount of lift on one wing by means of a spoiler, or by altering the lift on both wings by means of ailerons. This discussion will be limited to the use of ailerons as the means of lateral control.

Since the purpose of the ailerons is to create a rolling moment, a logical measure of aileron power would be the rolling moment created by a given aileron deflection. Before progressing to the actual development of this relationship, it is necessary to make several definitions. A positive deflection of either aileron,  $+\delta_a$ , is defined as one which produces a positive rolling moment, (right wing down). Thus, by definition,  $C_{l\delta_a}$  is positive. Also, in this discussion, total aileron deflection is defined as the sum of the two individual aileron deflections. Thus,

$$\delta_{aTotal} = \delta_{aLeft} + \delta_{aRight} \quad (5.72)$$

The assumption will be made that the wing cp shift due to aileron deflection will not alter the value of  $C_{l\beta}$ . The distance from the x-axis to the cp of the wing will be labeled  $\bar{Y}$ . When the ailerons are deflected, they produce a change in lift on both wings. This total change in lift,  $\Delta L$ , produces a rolling moment,  $\mathcal{L}$ .

$$\mathcal{L} = \Delta L \cdot \bar{Y} \quad (5.73)$$

$$\mathcal{L} = \frac{\partial C_{La}}{\partial \alpha_a} \cdot \Delta \alpha_a \cdot q_a \cdot S_a \cdot \bar{Y} \quad (5.74)$$

Where the "a" subscripts refer to "aileron" values.

$$\mathcal{L} = a_a \Delta \alpha_a q_a S_a \bar{Y} \quad (5.75)$$

$$C_l = \frac{a_a \Delta \alpha_a S_a \bar{Y}}{S_w b_w} \quad (5.76)$$

Where  $\Delta \alpha_a = \delta_{aTotal}$

$$C_l = \frac{a_a \delta_{aTotal} S_a \bar{Y}}{S_w b_w} \quad (5.77)$$

$$\frac{\partial C_l}{\partial \delta_a} = C_{l\delta_a} = \frac{a_a S_a \bar{Y}}{b_w S_w} \quad (5.78)$$

Thus, from equation 5.78, it can be seen that lateral control power is a function of the aileron airfoil section, the area of the aileron in relation to the area of the wing, and the location of the wing cp.

## 5.14 IRREVERSIBLE CONTROL SYSTEMS

Now that both  $C_{l\beta}$  and  $C_{l\delta_a}$  have been discussed, it is possible to develop a parameter which can be measured in flight to determine the static lateral stability of an aircraft. As in the directional case, the maneuver that will be flown will be a steady straight sideslip. Considering this maneuver, equation 5.6 reduces to,

$$C_l = C_{l\beta}\beta + C_{l\delta_a}\delta_a + C_{l\delta_r}\delta_r = 0 \quad (5.79)$$

$$\delta_a = -\frac{C_{l\beta}\beta}{C_{l\delta_a}} - \frac{C_{l\delta_r}}{C_{l\delta_a}}\delta_r \quad (5.80)$$

$$\frac{\partial \delta_a(\text{Fixed})}{\partial \beta} = -\frac{C_{l\beta}(\text{Fixed})}{C_{l\delta_a}} \quad (5.81)$$

Thus, since  $C_{l\delta_a}$  is known once the aircraft is built,  $\partial \delta_a / \partial \beta$ , can be taken as a direct measure of the static lateral stability of an aircraft. Again, the subscript "Fixed" has been added as a reminder that in this development the aileron has not been free to "float."

Equation 5.81 reveals that for static lateral stability, a plot of  $\partial \delta_a / \partial \beta$  should have a positive slope. Refer to figure 5.34.

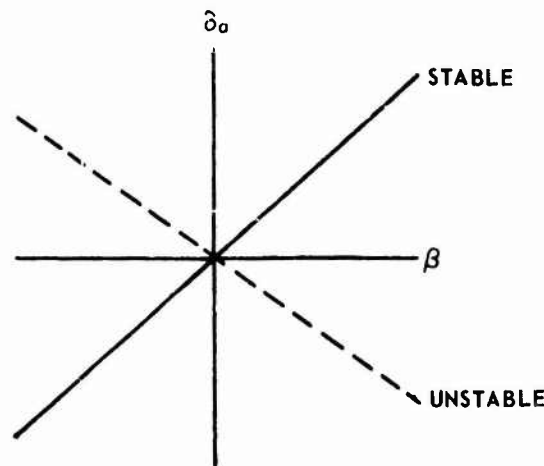


Figure 5.34 AILERON DEFLECTION VERSUS SIDESLIP ANGLE

## 5.15 REVERSIBLE CONTROL SYSTEMS

It is now necessary to consider an aircraft with a reversible control system. On this type aircraft, the ailerons are free to float in response to their hinge moments. Using the same approach as in the directional case, it is possible to derive an expression that will relate the "Aileron Free" static lateral stability to parameters that can be easily measured in flight.

In a steady straight sideslip,  $\mathcal{L} = 0$ . Therefore, it follows that  $\Sigma \mathcal{L}_{\text{Aileron Hinge Pin}} = 0$ . Now if moments are summed about the aileron hinge pin,

the aileron force exerted by the pilot,  $F_a$ , acts through a moment arm and gearing mechanism, both accounted for by some constant,  $K$ , and must balance the other aerodynamic rolling moments so that  $\Sigma \mathcal{L}_{\text{Aileron Hinge Pin}} = 0$ . Thus, in steady straight flight,

$$\Sigma \mathcal{L}_{\text{Aileron Hinge Pin}} = 0 = F_a \cdot K + H_a \quad (5.82)$$

$$F_a = -G \cdot H_a \quad (5.83)$$

Where  $G$  is merely  $1/K$ .

Knowing,

$$H_a = C_h q_a S_a c_a \quad (5.84)$$

From equation 5.26

$$H_a = q_a S_a c_a (C_{h_{\alpha_a}} \cdot \alpha_a + C_{h_{\delta_a}} \cdot \delta_a) \quad (5.85)$$

Thus, equation 5.83 becomes,

$$F_a = -G q_a S_a c_a (C_{h_{\alpha_a}} \cdot \alpha_a + C_{h_{\delta_a}} \cdot \delta_a) \quad (5.86)$$

From equation 5.27,

$$C_{h_{\alpha_a}} \cdot \alpha_a = -C_{h_{\delta_a}} \cdot \delta_{a(\text{Float})} \quad (5.87)$$

Equation 5.86 becomes,

$$F_a = -G q_a S_a c_a (-C_{h_{\delta_a}} \cdot \delta_{a(\text{Float})} + C_{h_{\delta_a}} \cdot \delta_a) \quad (5.88)$$

$$F_a = -G q_a S_a c_a C_{h_{\delta_a}} (\delta_a - \delta_{a(\text{Float})}) \quad (5.89)$$

The difference between where the pilot pushes the aileron,  $\delta_a$ , and the amount it floats,  $\delta_{a(\text{Float})}$ , is the free position of the aileron,  $\delta_{a(\text{Free})}$ .

Therefore,

$$F_a = -G q_a S_a c_a C_{h\delta_a} \delta_{a(\text{Free})} \quad (5.90)$$

$$\frac{\partial F_a}{\partial \beta} = -G q_a S_a c_a C_{h\delta_a} \frac{\partial \delta_{a(\text{Free})}}{\partial \beta} \quad (5.91)$$

From equation 5.81, it can be shown that,

$$\frac{\partial \delta_{a(\text{Free})}}{\partial \beta} = - \frac{C_{l\beta}(\text{Free})}{C_{l\delta_a}} \quad (5.92)$$

Thus,

$$\frac{\partial F_a}{\partial \beta} = G q_a S_a c_a \frac{C_{h\delta_a}}{C_{l\delta_a}} C_{l\beta}(\text{Free}) \quad (5.93)$$

This equation shows that the parameter  $\partial F_a / \partial \beta$ , can be taken as an indication of the aileron free static lateral stability of an aircraft. This parameter can be readily measured in flight.

An analysis of equation 5.93 reveals that for stable dihedral effect, a plot of  $\partial F_a / \partial \beta$  would have a positive slope. Refer to figure 5.35.

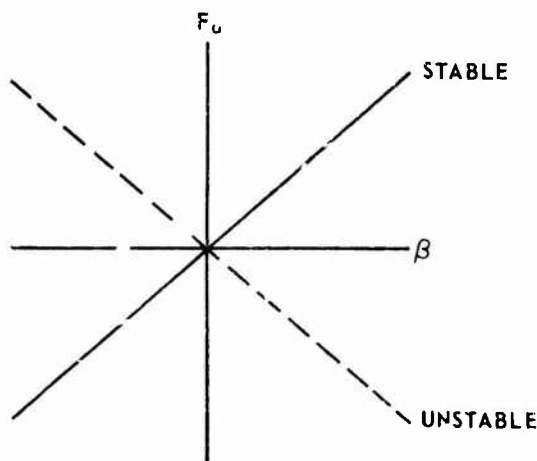


Figure 5.35 AILERON FORCE VERSUS SIDESLIP ANGLE

## 5.16 ROLLING PERFORMANCE

It has been shown how aileron force and aileron deflection can be used as a measure of the stable dihedral effect of an aircraft. However, it is now necessary to consider how aileron force and aileron deflection affect the rolling capability of the aircraft. For example, full aileron deflection may produce excellent rolling characteristics on certain aircraft, however, because of the large aileron forces required, the pilot may not be able to fully deflect the ailerons, thus making the overall rolling performance unsatisfactory. Thus, it is necessary to evaluate the rolling performance of the aircraft.

The rolling qualities of an aircraft can be evaluated by examining the parameters  $F_a$ ,  $\delta_a$ ,  $p$  and  $(pb/2V)$ . Although the importance of the first three parameters is readily apparent, the parameter  $(pb/2V)$  needs some additional explanation. Physically,  $(pb/2V)$  may be described as the helix angle that the wing tip of a rolling aircraft describes. Refer to figure 5.36.

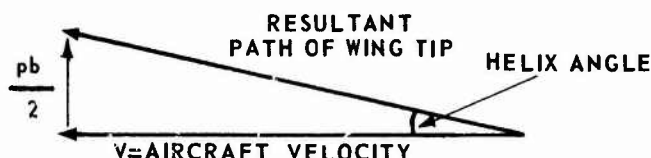


Figure 5.36  
WING TIP HELIX ANGLE

It can be seen that,

$$\tan (\text{Helix Angle}) = \frac{pb}{2V} \quad (5.94)$$

Assuming a small angle,

$$\text{Helix Angle} = \frac{pb}{2V} \quad (5.95)$$

Figure 5.36 is a vectorial presentation of the wind forces acting on the downgoing wing during a roll. It shows that the angle of attack of the downgoing wing is increased due to roll rate. Thus  $(pb/2V)$  represents a damping term.

With the foregoing background, it is possible to discuss the effect of the parameters,  $F_a$ ,  $\delta_a$ ,  $p$ ,  $(pb/2V)$  throughout the flight envelope of the aircraft.

From equation 5.90, it can be seen that

$$F_a = (f) V^2 \delta_a(\text{Free}) \quad (5.96)$$

$$\delta_a(\text{Free}) = (f) F_a \frac{1}{V^2} \quad (5.97)$$

To derive a functional relationship for  $(pb/2V)$ , it is necessary to start with,

$$C_l = C_{l_\beta} \beta + C_{l_{\dot{\beta}}} \dot{\beta} + C_{n_p} \hat{p} + C_{n_r} \hat{r} + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r \quad (5.6)$$

and examine the effects of roll terms only, i.e., assume that the roll moment developed is due to the interaction of moments due to  $\delta_a$  and roll damping only. Therefore, equation 5.6 becomes,

$$C_l = C_{l_p} \hat{p} + C_{l_{\delta_a}} \delta_a = C_{l_p} \left(\frac{pb}{2V}\right) + C_{l_{\delta_a}} \delta_a \quad (5.98)$$

Below Mach or aerolastic effects,  $C_{l_{\text{Max}}} = \text{constant}$ , so if it is desired to evaluate an aircraft's maximum rolling performance, equation 5.98 becomes,

$$C_{l_p} \left(\frac{pb}{2V}\right) + C_{l_{\delta_a}} \delta_a = \text{constant} \quad (5.99)$$

$$\left(\frac{pb}{2V}\right) = \frac{\text{Constant} - C_{l_{\delta_a}} \delta_a}{C_{l_p}} \quad (5.100)$$

$$\left(\frac{pb}{2V}\right) = (f) \delta_a \quad (5.101)$$

From equation 5.97,

$$\left(\frac{pb}{2V}\right) = (f) \delta_a = (f) F_a \frac{1}{V^2} \quad (5.102)$$

A function relationship for roll rate,  $p$ , can be derived from equation 5.100,

$$p = \frac{\text{Constant} - C_{l_{\delta_a}} \delta_a}{C_{l_p}} \cdot \frac{2}{b} \cdot V \quad (5.103)$$

$$p = (f) V \delta_a \quad (5.104)$$

From equation 5.97,

$$p = (f) V \delta_a = (f) F_a \frac{1}{V} \quad (5.105)$$

To summarize, the rolling performance of an aircraft can be evaluated by examining the parameters,  $F_a$ ,  $\delta_a$ ,  $p$ , and  $(pb/2V)$ . Functional relationships have been developed in order to look at the variance of these parameters below Mach or aeroelastic effects. These functional relationships are:

$$F_a = (f) V^2 \delta_a \quad (5.96)$$

$$\delta_a = (f) F_a \frac{1}{V^2} \quad (5.97)$$

$$\left(\frac{pb}{2V}\right) = (f) \delta_a = (f) F_a \frac{1}{V^2} \quad (5.102)$$

$$p = (f) V \delta_a = (f) F_a \frac{1}{V} \quad (5.105)$$

These relationships are expressed graphically in figure 5.37 for a case in which the pilot desires the maximum roll rate at all airspeeds.

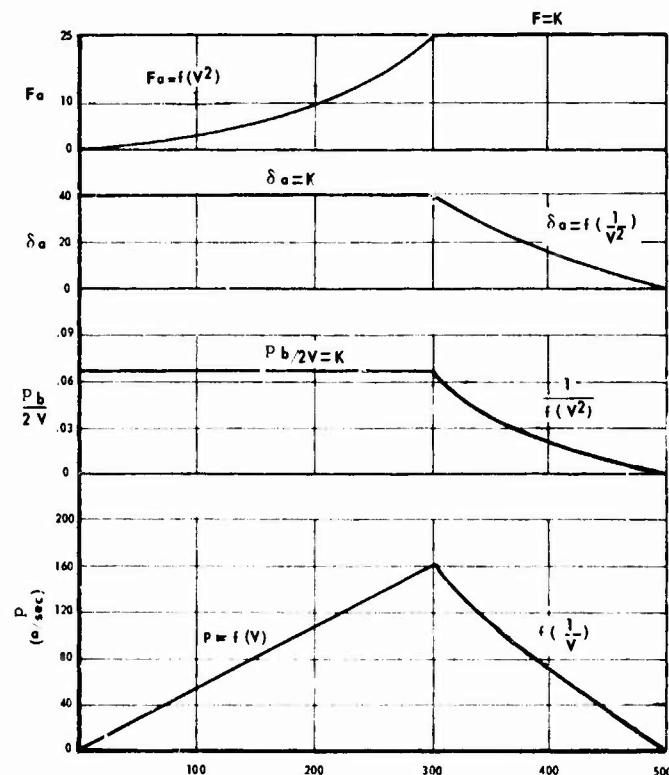


Figure 5.37 ROLLING PERFORMANCE

As indicated in equation 5.96, the force required to hold a constant aileron deflection will vary as the square of the airspeed. The force required by the pilot to hold full aileron deflection will increase in this manner until the aircraft reaches  $V_{Max}$  or until the pilot is unable to apply any more force. In figure 5.37, it is assumed that the pilot can supply a maximum of 25 pounds force and that this force is reached at 300 knots. If the speed is increased further, the aileron force will remain at this 25 pound maximum value. The curve of aileron deflection versus airspeed shows that the pilot is able to maintain full aileron deflection out to 300 knots. Inspection of equation 5.97 shows that if aileron force is constant beyond 300 knots, then aileron deflection will be proportional to  $(1/V^2)$ . Equation 5.102 shows that  $(pb/2V)$  will vary in the same manner as aileron deflection. Inspection of equation 5.105 shows that the maximum roll rate available will increase linearly as long as the pilot can maintain maximum aileron deflection; up to 300 knots in this case. Beyond this point, the maximum roll rate will fall off hyperbolically. That is, above 300 knots,  $p$  is proportional to  $1/V$ . It follows, then, that at high speeds the maximum roll rate may become unacceptably low. One method of combating this problem is to increase the pilot's mechanical advantage by adding boosted or fully powered ailerons.

#### ■ 5.17 ROLL DAMPING $C_{lp}$

Aircraft roll damping comes from the wing and the vertical tail. The algebraic sign of  $C_{lp}$  is negative as long as the local angle of attack remains below the local stall angle of attack.

The wing contribution to  $C_{lp}$  arises from the change in wing angle of attack that results from the rolling velocity. It has already been shown that the downgoing wing in a rolling maneuver experiences an increase in angle of attack and that this increased  $\alpha$  tends to develop a rolling moment that opposes the original rolling moment. However, when the wing is near the aerodynamic stall, a rolling motion may cause the downgoing wing to exceed the stall angle of attack. In this case, the local lift curve slope may fall to zero or even reverse sign. The algebraic sign of the wing contribution to  $C_{lp}$  may then become positive. This is what occurs when a wing "autorotates," as in spinning.

The vertical tail contribution to  $C_{lp}$  arises from the fact that when the aircraft is rolled, the angle of attack on the vertical tail is changed. This change in angle of attack develops a lift force. If the vertical tail cg is above or below the aircraft cg, the rolling moment developed will oppose the original rolling moment and  $C_{lp}$  due to a conventional vertical tail or a ventral fin will be negative.

#### ■ 5.18 ROLLING MOMENT DUE TO YAW RATE - $C_r$

The contributions to this derivative come from two sources, the wings and the vertical tail.

As the aircraft yaws, the velocity of the relative wind is increased on the outboard wing and decreased on the inboard wing. This causes the

outboard wing to produce more lift and thus produces a rolling moment. A right yaw would produce more lift on the left wing and thus a rolling moment to the right. Thus, the algebraic sign of the wing contribution to  $C_{l_r}$  is positive.

The tail contribution to  $C_{l_r}$  arises from the fact that as the aircraft is yawed, the angle of attack on the vertical tail is changed. Refer to figure 5.38.

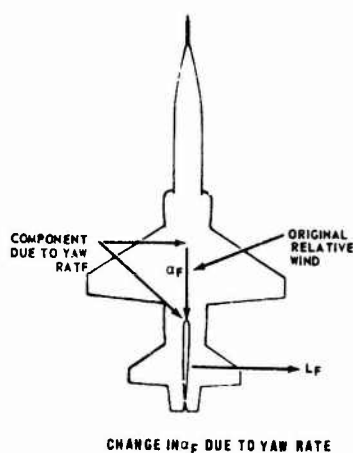


Figure 5.38

The lift force thus produced,  $L_f$ , will create a rolling moment if the vertical tail cg is above or below the cg. For a conventional vertical tail, the sign of  $C_{l_r}$  will be positive while for a ventral fin the sign will be negative.

#### ■ 5.19 ROLLING MOMENT DUE TO RUDDER DEFLECTION -- $C_{l_r}$

When the rudder is deflected, it creates a lift force on the vertical tail. If the cg of the vertical tail is above or below the aircraft cg a rolling moment will result. Refer to figure 5.39.

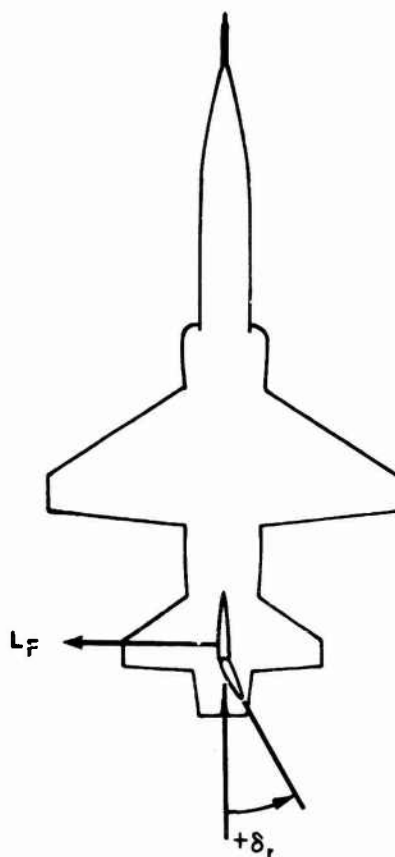


Figure 5.39 LIFT FORCE DEVELOPED AS A RESULT OF  $\delta_r$

It can be seen that if the cp of the vertical tail is above the cg, as with a conventional vertical tail, the sign of  $C_{l_{\delta_r}}$  will be negative. However, with a ventral fin, the sign would be positive.

It is interesting to note that the effects of  $C_{l_{\delta_r}}$  and  $C_{l_{\beta}}$  are opposite in nature. When the rudder is deflected to the right, initially, a rolling moment to the left is created due to  $C_{l_{\delta_r}}$ . However, as sideslip develops due to the rudder deflection, dihedral effect,  $C_{l_{\beta}}$ , comes into play and causes a resulting rolling moment to the right. Therefore, when a pilot applies right rudder to pick up a left wing, he initially creates a rolling moment to the left and finally, to the right.

## 5.20 ROLLING MOMENT DUE TO LAG EFFECTS IN SIDEWASH - $C_{l\dot{\beta}}$

In the discussion of  $C_{n\dot{\beta}}$ , it was pointed out that during an increase in  $\beta$ , the angle of attack of the vertical tail will be less than it will finally be in steady state conditions. If the cp of the vertical tail is displaced from the aircraft cg, this change in  $\alpha_F$  due to lag effects will alter the rolling moment created during the  $\beta$  build up period. Because of lag effects,  $C_{l\dot{\beta}}$  will be less during the  $\beta$  build up period than at steady state. Thus, for a conventional vertical tail, the algebraic sign of  $C_{l\dot{\beta}}$  is positive.

Again, it should be pointed out that there is widespread disagreement over the interpretation of data concerning lag effects in sidewash and that the foregoing is only one basic approach to a many faceted and complex problem.

## 5.21 HIGH SPEED CONSIDERATIONS OF STATIC LATERAL STABILITY

$C_{l\beta}$  - Generally,  $C_{l\beta}$  is not greatly affected by Mach number. However, in the transonic region the increase in the lift curve slope of the vertical tail increases this contribution to  $C_{l\beta}$  and usually results in an overall increase in  $C_{l\beta}$  in the transonic region.

$C_{l\dot{\beta}}$  - Because of the decrease in the lift curve slope of all aerodynamic surfaces in supersonic flight, lateral control power decreases as Mach number increases supersonically.

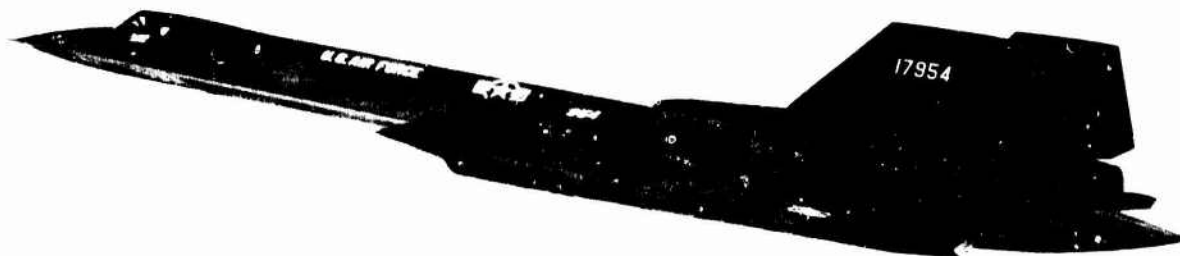
Aeroelasticity problems have been quite predominant in the lateral control system, since in flight at very high dynamic pressures the wing torsional deflections which occur with aileron usage are considerable and cause noticeable changes in aileron effectiveness. At some high dynamic pressures, dependent upon the given wing structural integrity, the twisting deformation might be great enough to nullify the effect of aileron deflection and the aileron effectiveness will be reduced to zero. Since at speeds above the point where this phenomenon occurs, rolling moments are created which are opposite in direction to the control deflection, this speed is termed "aileron reversal speed." In order to alleviate this characteristic the wing must have a high torsional stiffness which presents a significant design problem in sweptwing aircraft. For an aircraft design of the B-47 type, it is easy to visualize how aeroelastic distortion might result in a considerable reduction in lateral control capability at high speeds. In addition, lateral control effectiveness at transonic Mach numbers may be reduced seriously by flow separation effects as a result of shock formation. However, modern high speed fighter designs have been so successful in introducing sufficient rigidity into wing structures and employing such design modifications as split ailerons, inboard ailerons, spoiler systems, etc., that the resulting high control power, coupled with the low  $C_{l\dot{\beta}}$  of low aspect ratio planforms, has resulted in the lateral control becoming an accelerating device rather than a rate control. That is to say, a steady state rolling velocity is normally not reached prior to attaining the desired bank angle. Consequently, many high speed aircraft have a type of differential aileron system to provide the pilot with much more control surface during approach and landings and to restrict his degree of control in other areas of flight.

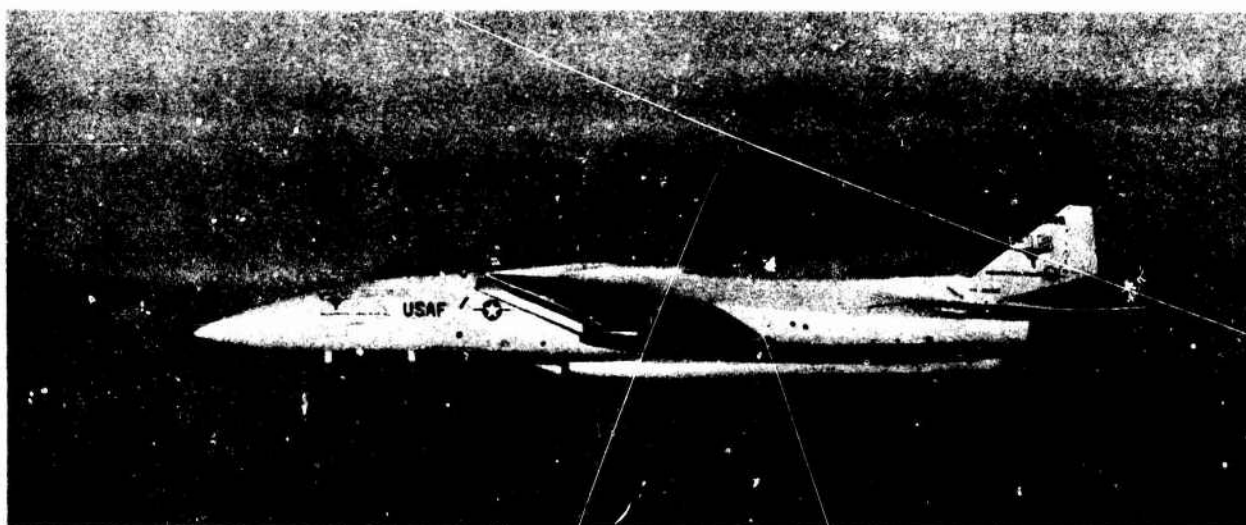
Spoiler controls are quite effective in reducing aeroelastic distortions since the pitching moment changes due to spoilers are generally smaller than those for a flap type control surface. However, a problem associated with spoilers is their tendency to reverse the roll direction for small stick inputs during transonic flight. This occurs as a result of re-energizing the boundary layer by a vortex generator effect for very small deflections of the spoiler, which can reduce the magnitude of the shock induced separation and actually increase the lift on the wing. This difficulty can be eliminated by proper design techniques.

$C_{l_p}$  - Since the development of "damping" requires the development of lift on either the wing or the tail, it is dependent on the value of the lift curve slope. Thus, as the lift curve slope of both the wing and tail decrease supersonically,  $C_{l_p}$  will decrease. Also, since most supersonic designs make use of low aspect ratio surfaces,  $C_{l_p}$  will tend to be less for these designs.

$C_{l_r}$  and  $C_{l_{\delta r}}$  - Both of these derivatives depend on the development of lift and will decrease as the lift curve slope decreases supersonically.

$C_{l_{\beta}}$  - Data on the supersonic variation of this derivative is sketchy, but it probably will not change significantly with Mach number.





## CHAPTER 6 DYNAMICS

### ■ 6.1 INTRODUCTION

REVISED DECEMBER 1973

The study of dynamics is concerned with the time history of the motion of some physical system. An aircraft is such a system, and its equations of motion can be derived from theory. In their basic form these equations comprise a set of six simultaneous, nonlinear differential equations with ill-defined forcing functions such as  $F_x = f$  (Aero, Gravity, Thrust). Recall from Chapter 1 that two methods were used to get these equations into a set of workable simultaneous linear differential equations:

1. Small perturbations were assumed such that products of perturbations were negligible.
2. Forcing functions were approximated by the linear part of a Taylor Series expansion for the forcing function.

This set of linear differential equations can then be operated on by Laplace transforms so that simple algebraic solutions followed by inverse transformations back to the time domain result in equations which describe the aircraft's motion as a function of time.

In the good old days when aircraft were simple, all aircraft exhibited the five characteristic dynamic modes of motion, two longitudinal and three lateral-directional modes. The two longitudinal modes are the short period and the phugoid; the three lateral-directional modes are the Dutch roll, the spiral, and the roll mode. Theoretical solutions for these modes of motion can be obtained by the methods listed above.

As aircraft control systems have increased in complexity, it is conceivable that one or more of these modes may not exist as a dominant longitudinal or lateral-directional mode. It can be expected, however, that frequently the higher order effects of complex control systems will be quick to die out and will leave the basic five dynamic modes of motion. When this is not the case, the development of special procedures may be required to meaningfully describe an aircraft's dynamic motion. For the purposes of this chapter, aircraft will be assumed to possess these five basic modes of motion.

During this study of aircraft dynamics, the solutions to both first order and second order systems will be of interest, and several important descriptive parameters will be used to define either a first order system or a second order system.

The quantification of handling qualities, that is, specify how the magnitude of some of these descriptive parameters can be used to indicate how well an aircraft can be flown, has been an extensive investigation which is by no means complete. Flight test, simulators, variable stability aircraft, engineering know-how, and pilot opinion surveys have all played major roles in this investigation. The military specification on aircraft handling qualities, MIL-F-8785B, is the culmination of this effort and has the intent of insuring that an aircraft will handle well if compliance has been achieved. This chapter will not attempt to evaluate how satisfactory MIL-F-8785B is for this purpose, but suffice it to say that even this comprehensive document has some room for improvement.

## ■ 6.2 DYNAMIC STABILITY

When it is necessary to investigate the dynamic stability characteristics of a physical system, the time history of its motion must be known.



As indicated earlier, this time history can be obtained theoretically and with good accuracy in many cases, depending on the depth of the theoretical analysis.

A particular mode of an aircraft's motion is defined to be "dynamically stable" if the parameters of interest tend toward finite values as time increases without limit. Some examples of dynamically stable time histories and some terms used to describe them are shown in figures 6.1 and 6.2.

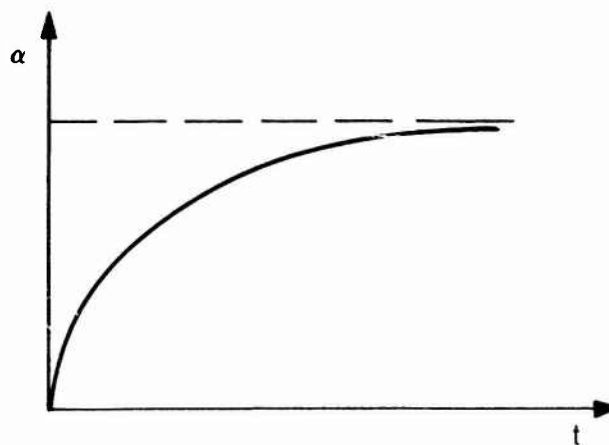


Figure 6.1 Exponentially Decreasing

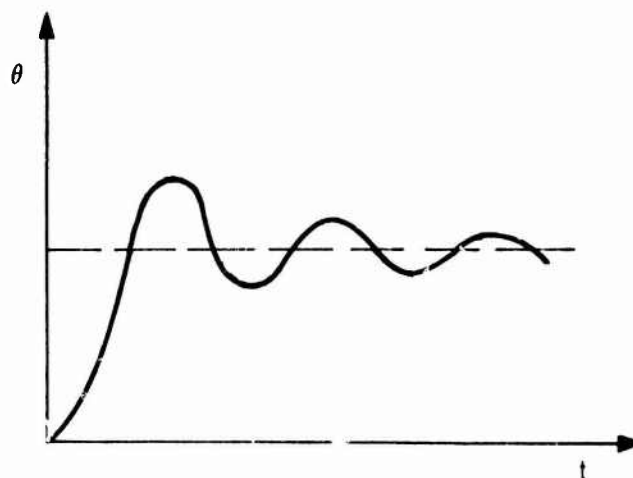
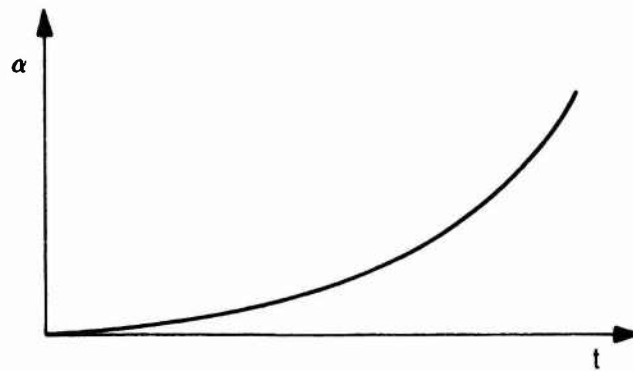
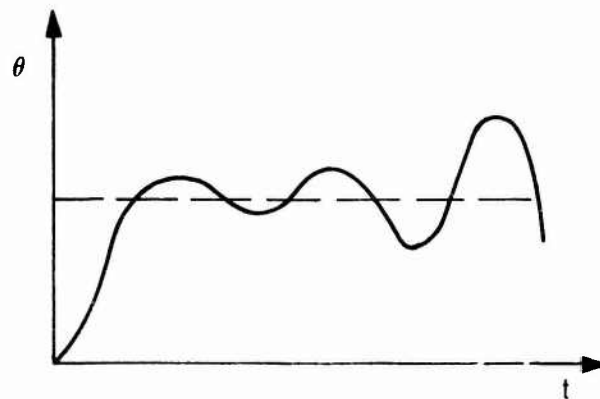


Figure 6.2 Damped Sinusoidal Oscillation

A mode of motion is defined to be "dynamically unstable" if the parameters of interest increase without limit as time increases without limit. Some examples of dynamic instability are shown in figures 6.3 and 6.4.

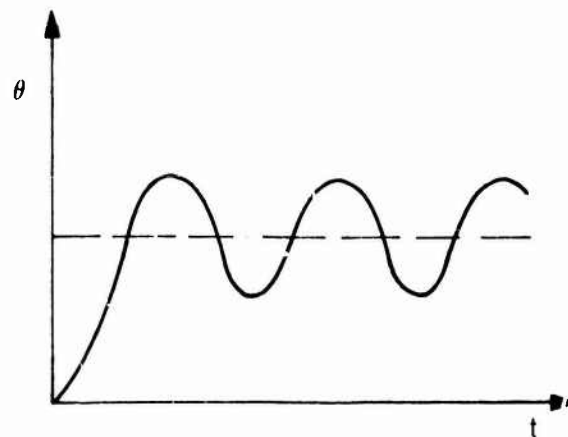


**Figure 6.3 Exponentially Increasing**



**Figure 6.4 Divergent Sinusoidal Oscillation**

A mode of motion is said to have "neutral dynamic stability" if the parameters of interest exhibit an undamped sinusoidal oscillation as time increases without limit. A sketch of such motion is shown in figure 6.5.



**Figure 6.5 Undamped Oscillation**

### ● 6.2.1 EXAMPLE PROBLEM

To emphasize the difference between static stability and dynamic stability the simple physical system shown in figure 6.6 consisting of a mass and a spring will be examined for both static stability and dynamic stability.

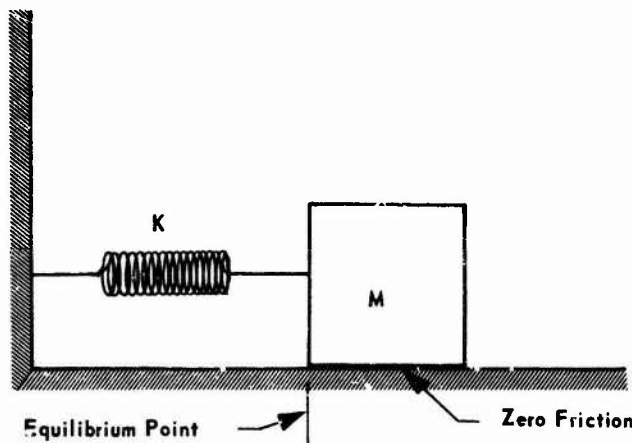


Figure 6.6

#### ● 6.2.1.1 STATIC STABILITY ANALYSIS

If the mass were displaced from its equilibrium position, then a spring force would exist to return  $M$  toward its initial position. Thus, this physical system has positive static stability.

#### ● 6.2.1.2 DYNAMIC STABILITY ANALYSIS

The motion of the system as a function of time must be known to describe its dynamic stability. Two methods could be used to find the time history of the motion of the block:

1. A test could be devised to perturb the block from its equilibrium position and the resulting motion would be observed for analysis.
2. If a good enough mathematical model of the system could be obtained, the equation of motion could be analyzed to describe its dynamic stability.

Using the theoretical approach, the equation of motion for this physical system is

$$x(t) = C_1 \cos \left( \sqrt{\frac{K}{M}} t + \phi \right)$$

where  $C_1$  and  $\phi$  are constants dependent on the initial velocity and displacement of the mass from its equilibrium condition. Examination of its equation of motion shows that this system has "neutral dynamic stability."

### ● 6.2.2 EXAMPLE PROBLEM

A similar analysis must be accomplished to analyze the aircraft shown in figure 6.7 for longitudinal static stability and dynamic stability. This aircraft is operating at a constant  $\alpha_0$  in 1-g flight.

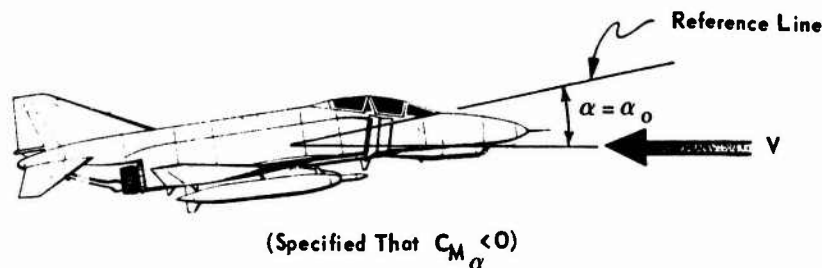


Figure 6.7

#### ● 6.2.2.1 STATIC STABILITY ANALYSIS

If the aircraft were displaced from its equilibrium flight conditions by increasing the angle of attack to  $\alpha = \alpha_0 + \Delta\alpha$ , then the change in pitching moment due to the increase in angle of attack would be nose down because  $C_{M_\alpha} < 0$ . Thus the aircraft has positive static longitudinal stability.

#### ● 6.2.2.2 DYNAMIC STABILITY ANALYSIS

The motion of the aircraft as a function of time must be known to describe its dynamic stability. Two methods could be used to find the time history of the motion of the aircraft:

1. A flight test could be flown in which the aircraft is perturbed from its equilibrium condition and the resulting motion is recorded and observed.
2. Solutions to the aircraft equations of motion could be obtained and analyzed.

A sophisticated solution to the aircraft equations of motion with valid aerodynamic inputs can result in good theoretically obtained time histories. However, the fact remains that the only way to discover the aircraft's actual dynamic motion is to flight test and record its motion for analysis.

### ■ 6.3 EXAMPLES OF FIRST AND SECOND ORDER DYNAMIC SYSTEMS

#### ● 6.3.1 SECOND ORDER SYSTEM WITH POSITIVE DAMPING

The problem of finding the motion of the block shown in figure 6.8 encompasses many of the methods and ideas that will be used in finding the time history of an aircraft's motion from its equations of motion.

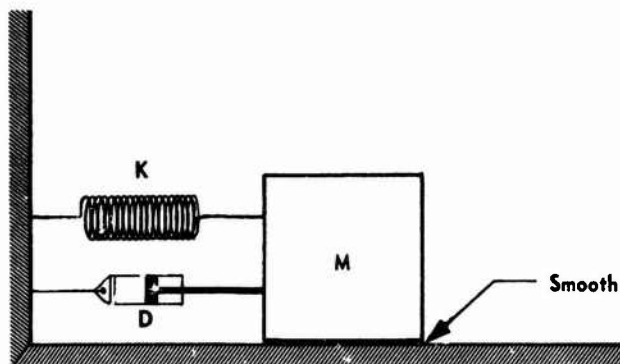


Figure 6.8 Second Order System

The differential equation of motion for this physical system is

$$0 = M \ddot{x} + D \dot{x} + K x \quad (6.1)$$

After Laplace transforming equation 6.1 and solving for  $X(S)$ , the displacement of the block in the Laplace domain, the result is

$$X(S) = \frac{Sx_0 + \dot{x}_0 + \frac{D}{M} x_0}{S^2 + \frac{D}{M} S + \frac{K}{M}} \quad (6.2)$$

The denominator of the  $S$  domain equation which gives the response of a system will be referred to as its "characteristic equation," and the symbol  $\Delta(S)$  will be used to indicate the characteristic equation.

The  $\Delta(S)$  of a second order system will frequently be written in a standard notation

$$0 = S^2 + 2 \zeta \omega_n S + \omega_n^2 \quad (6.3)$$

where

$\omega_n$  = natural frequency

$\zeta$  = damping ratio

The two terms natural frequency and damping ratio are frequently used to characterize the motion of second order systems.

Also, knowing the location of the roots of  $\Delta(S)$  on the complex plane makes it possible to immediately specify and sketch the dynamic motion associated with a system. Continuing to discuss the problem shown in figure 6.8 and making an identity between equations 6.2 and 6.3 results in

$$\omega_n = \sqrt{\frac{K}{M}} \quad (6.4)$$

$$\zeta = \frac{D}{2M\sqrt{\frac{K}{M}}} \quad (6.5)$$

The roots of  $\Delta(S)$  can be found from equation 6.3 to be

$$s_{1,2} = -\zeta\omega_n \pm i\omega_d \quad (6.6)$$

Where

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

Note that if  $(-1 < \zeta < 1)$ , then the roots of  $\Delta(S)$  comprise a complex conjugate pair, and for positive  $\zeta$  would result in root locations as shown in figure 6.9.

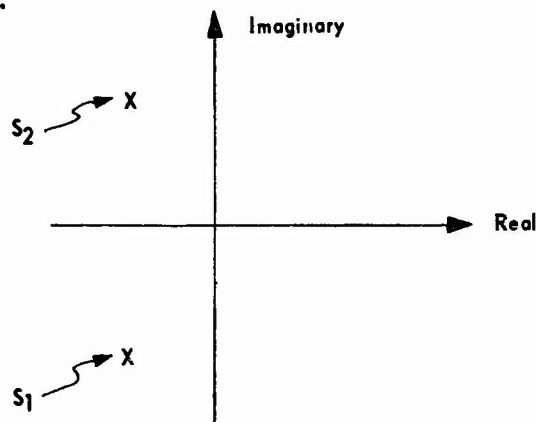


Figure 6.9 Complex Plane

The equation describing the time history of the block's motion can be written by knowing the roots of  $\Delta(S)$  given in equation 6.6.

$$x(t) = C_1 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \quad (6.7)$$

Where

$C_1$  and  $\phi$  are constants determined by boundary conditions.

Knowing either the  $\Delta(S)$  root location shown in figure 6.9 or equation 6.7 makes it possible to sketch or describe the time history of the motion of the block. The motion of the block shown in figure 6.8 as a function of time is a sinusoidal oscillation within an exponentially decaying envelope and is dynamically stable.

### 6.3.2 SECOND ORDER SYSTEM WITH NEGATIVE DAMPING

A similar procedure to that used in Section 6.3.1 can be used to find the motion of the block shown in figure 6.10.

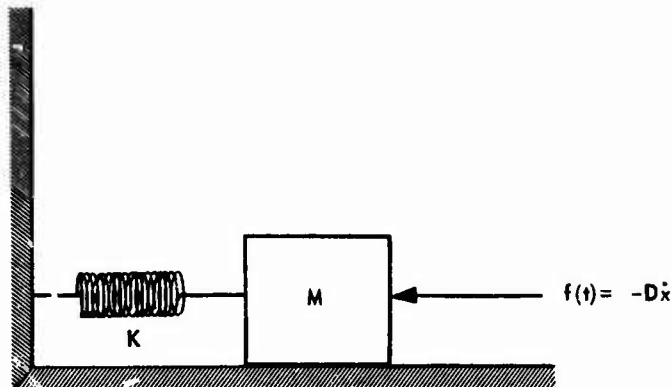


Figure 6.10

The differential equation of motion for this block is

$$0 = M \ddot{x} - D \dot{x} + K x \quad (6.8)$$

and the equation for  $X(S)$  is

$$X(S) = \frac{Sx_0 + \dot{x}_0 - \frac{D}{M} x_0}{S^2 - \frac{D}{M} S + \frac{K}{M}} \quad (6.9)$$

By inspection, for this system

$$\omega_n = \sqrt{\frac{K}{M}} \quad (6.10)$$

$$\zeta = \frac{-D}{2M\sqrt{\frac{K}{M}}} \quad (6.11)$$

From equation 6.11 note that the damping ratio has a negative value. The equation giving the time response of this system is

$$x(t) = C_1 e^{(\text{pos. value})t} \sin(\omega_d t + \phi) \quad (6.12)$$

where

$$-\zeta\omega_n = \text{pos. value}$$

For the range  $(-1 < \zeta < 0)$ , the roots of  $\Delta(S)$  for this system could again be plotted on the complex plane from equation 6.6 as shown in figure 6.11.

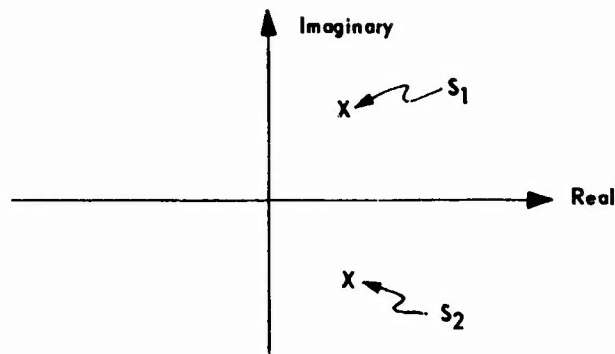


Figure 6.11 Complex Plane

The motion of this system can now be sketched or described. The motion of this system is a sinusoidal oscillation within an exponentially diverging envelope and is dynamically unstable.

### ● 6.3.3 UNSTABLE FIRST ORDER SYSTEM

Assume that some physical system has been mathematically modeled and its equation of motion in the  $S$  domain is

$$\phi(S) = \frac{0.5}{0.4S - 0.7} \quad (6.13)$$

For this system the characteristic equation is

$$\Delta(S) = S - 1.75$$

And its root is shown plotted on the complex plane in figure 6.12.

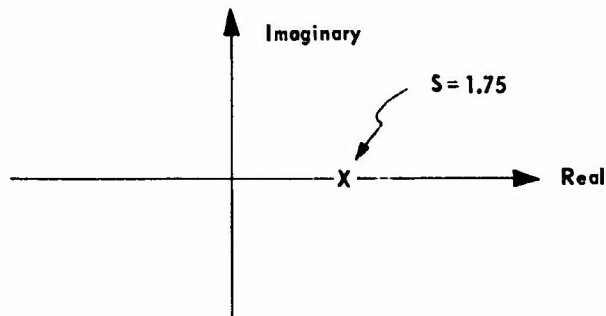


Figure 6.12 Complex Plane

The equation of motion in the time domain becomes

$$\phi(t) = 1.25 e^{1.75 t} \quad (6.14)$$

Note that it is possible to sketch or describe the motion for this system by knowing the location of the root of  $\Delta(S)$  or its equation of motion.

For an unstable first order system such as this, one parameter that can be used to characterize its motion is  $T_2$ , defined as the time to double amplitude. Without proof,

$$T_2 = \frac{.693}{a} \quad (6.15)$$

For a first order system described by

$$z = C_1 e^{at} \quad (6.16)$$

Note that for a stable first order system a similar parameter has been defined:  $T_{1/2}$  is the time to half amplitude.

$$T_{1/2} = \frac{-0.693}{a} \quad (6.17)$$

Where the term  $a$  must have a negative value for a stable system.

#### ● 6.3.4 ADDITIONAL TERMS USED IN DYNAMICS

The time constant,  $\tau$ , is defined for a stable first order system as the time when the exponent of  $e$  in the system equation is  $-1$ . From equation 6.16,

$$\tau = \frac{-1}{a} \quad (6.18)$$

The time constant can be thought of as the time required for the parameter of interest to accomplish  $(1 - \frac{1}{e})$ th of its total value change. Note that

$$\frac{1}{e} = \frac{1}{2.718} \approx \frac{1}{3}$$

so that

$$(1 - \frac{1}{e}) \approx \frac{2}{3}$$

With this in mind, it is easy to visualize an approximate value for a system time constant from a time history. Thus, the magnitude of the time constant gives a measure of how quickly the dynamic motion of a first order system occurs.

The following list contains some terms commonly used to describe second order systems based on damping ratio values:

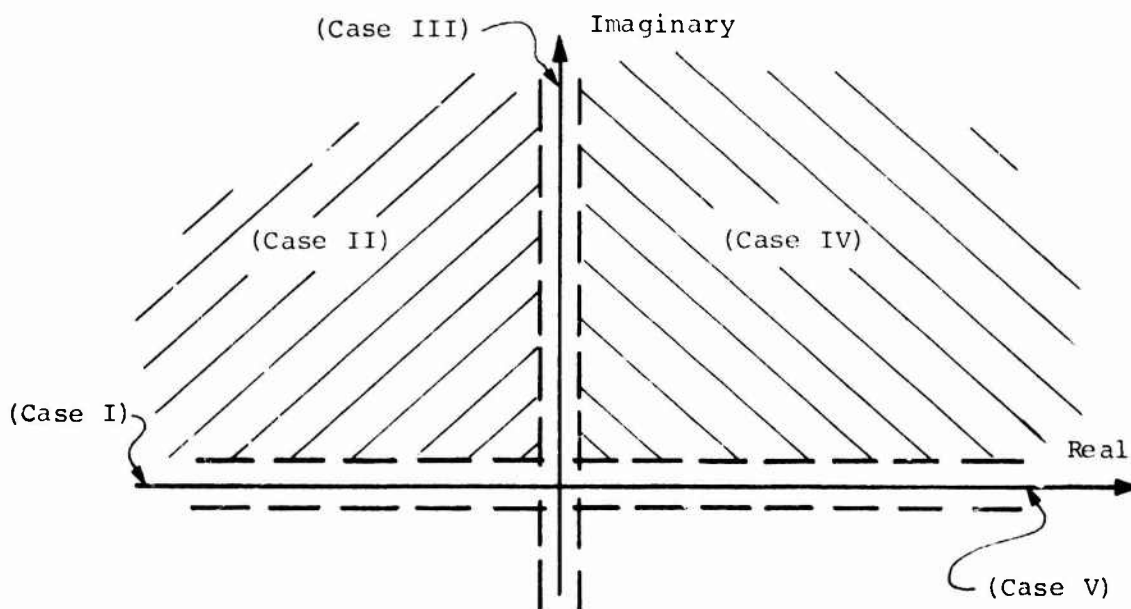
<u>Term</u>	<u>Damping Ratio Value</u>
Overdamped	$1 < \zeta$
Critically damped	$1 = \zeta$
Underdamped	$0 < \zeta < 1$
Undamped	$0 = \zeta$
Negatively damped	$\zeta < 0$

Typical responses can be visualized after the value of  $\zeta$  has been established.

#### 6.4 THE COMPLEX PLANE

It is possible to describe the type response a system will have by knowing the location of the roots of its characteristic equation on the complex plane. A first order response will be associated with each real root, and a complex conjugate pair will have a second order response that is either stable, neutrally stable, or unstable. A complicated system such as an aircraft might have a characteristic equation with several roots, and the total response of such a system will be the sum of the responses associated with each root. A summary of root location and associated response is presented in the following list and sketch below.

	<u>Root Location</u>	<u>Associated Response</u>
Case I	On the negative Real axis (1st Order Response)	Dynamically stable with exponential decay
Case II	In the left half plane off the negative Real axis (2nd Order Response)	Dynamically stable with sinusoidal oscillation in exponentially decaying envelope
Case III	On the Imaginary axis (2nd Order Response)	Neutral dynamic stability
Case IV	In the right half plane off the positive Real axis (2nd Order Response)	Dynamically unstable with sinusoidal oscillation in exponentially increasing envelope
Case V	On the positive Real axis (1st Order Response)	Dynamically unstable with exponential increase



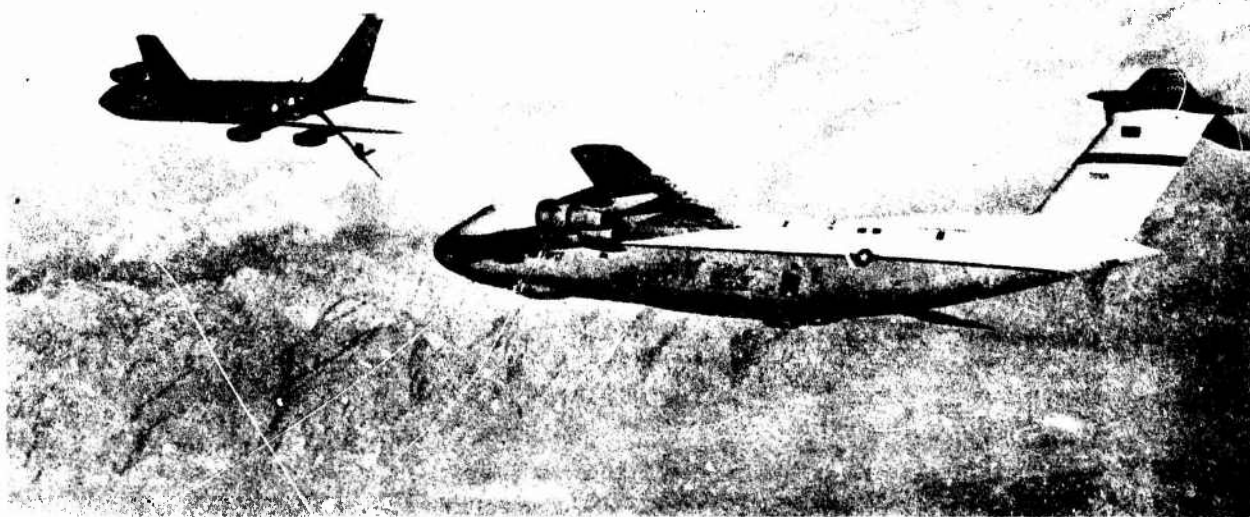
Note that any roots appearing in the upper half plane must have reflected roots in lower half plane.

## ■ 6.5 HANDLING QUALITIES

Because the "goodness" with which an aircraft flies is often stated as a general appraisal . . . "My F-69 is the best damn fighter ever built, and it can outfly and outshoot any other airplane." "It flies good." "That was really hairy." . . . you probably can understand the difficulty of measuring how well an aircraft handles. The basic question of what parameters to measure and how those parameters relate to good handling qualities has been a difficult one, and the total answer is not yet available. The current best answers for military aircraft are found in MIL-F-8785B, the specification for the "Flying Qualities of Piloted Airplanes."

When an aircraft is designed for performance, the design team has definite goals to work toward . . . a particular takeoff distance, a minimum time to climb, or a specified combat radius. If an aircraft is also to be designed to handle well, it is necessary to have some definite handling quality goals to work toward. Success in attaining these goals can be measured by flight tests for handling qualities when some rather firm standards are available against which to measure and from which to recommend.

To make it possible to specify acceptable handling qualities it was necessary to evolve some flight test measurable parameters. Flight testing results in data which yield values for the various handling quality parameters, and the military specification gives a range of values that should insure good handling qualities. Because MIL-F-8785 is not the ultimate answer, the role of the test pilot in making accurate qualitative observations and reports in addition to generating the quantitative data is of great importance in handling qualities testing.



One method that has been extensively used in handling qualities quantification is the use of pilot opinion surveys and variable stability aircraft. For example, a best range of values for the short period damping ratio and natural frequency could be identified by flying a particular aircraft type to accomplish a specific task while allowing the  $\zeta$  and  $\omega_n$  to vary. From the opinions of a large number of pilots, a valid best range of values for  $\zeta$  and  $\omega_n$  could thus be obtained, as shown in figure 6.13.

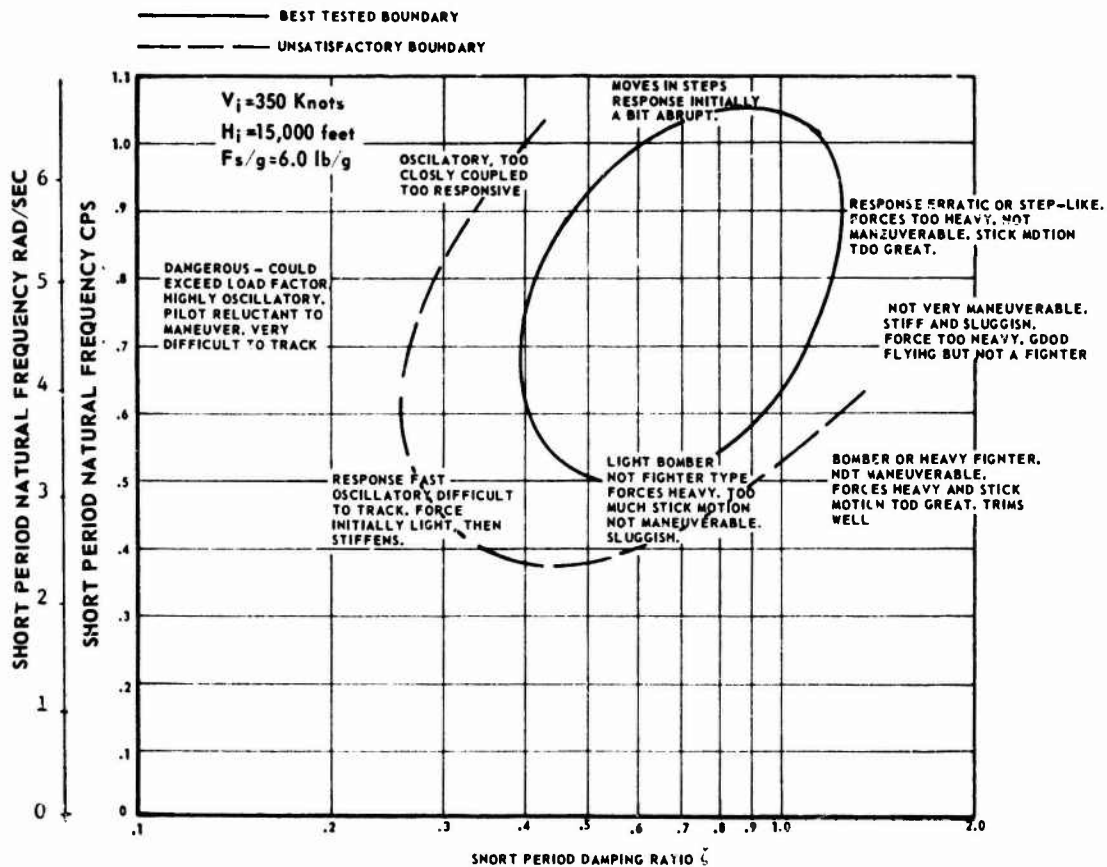


Figure 6.13

### ● 6.5.1 FREE RESPONSE

The  $\zeta$  and  $\omega_n$  being discussed here are the aircraft free response characteristics which describe aircraft motion without pilot inputs. With the pilot in the loop, the free response of the aircraft is hidden as pilot inputs are continually made. The closed loop block diagram shown in figure 6.14 should be used to understand aircraft closed loop and open loop response.

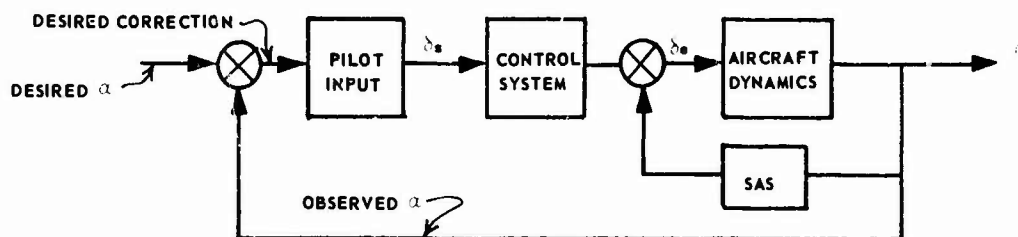


Figure 6.14

The free response of an aircraft does relate directly to how well the aircraft can be flown with a pilot in the loop, and many of the pertinent handling qualities parameters are for the open loop aircraft.

It must be kept in mind that the real test of an aircraft's handling qualities is how well it can be flown closed loop to accomplish a particular mission. Closed loop handling quality evaluations such as air-to-air tracking in a simulated air combat maneuvering mission play an important part of determining how well an aircraft handles.

### ● 6.5.2 PILOT RATING SCALES<sup>1</sup>

The Calspan Corporation (formerly Cornell Aeronautical Laboratory) has made notable contributions to the use and understanding of pilot rating scales and pilot opinion surveys. Except for minor variations between pilots, which sometimes prevent a sharp delineation between acceptable and unacceptable flight characteristics, there is very definite consistency and reliability in pilot opinion. In addition, the opinions of well qualified test pilots can be exploited because of their engineering knowledge and experience in many different aircraft types.

The stability and control characteristics of airplanes are generally established by wind tunnel measurement and by technical analysis as part of the airplane design process. The handling qualities of a particular airplane are related to the stability and control characteristics. The relationship is a complex one which involves the combination of the airplane and its human pilot in the accomplishment of the intended mission. It is important that the effects of specific stability and control characteristics be evaluated in terms of their ultimate effects on the suitability of the pilot-vehicle combination for the mission. On the basis of

<sup>1</sup>A Revised Pilot Rating Scale for the Evaluation of Handling Qualities, CAL Report No. 153, Robert P. Horper and George E. Cooper.

this information, intelligent decisions can be made during the airplane design phase which will lead to the desired handling qualities of the final product.

There are three general ways in which the relationship between stability and control parameters and the degree of suitability of the airplane for the mission may be examined:

1. Theoretical analysis
2. Experimental performance measurement
3. Pilot evaluation

Each of the three approaches has an important role in the complete evaluation. One might ask, however, why is the pilot assessment necessary? At present a mathematical representation of the human operator best lends itself to analysis of specific simple tasks. Since the intended use is made up of several tasks and several modes of pilot-vehicle behavior, difficulty is experienced first in accurately describing all modes analytically, and second in integrating the quality of the subordinate parts into a measure of overall quality for the intended use. In spite of these difficulties, theoretical analysis is fundamental to understanding pilot-vehicle difficulties, and pilot evaluation without it remains a purely experimental process.

The attainment of satisfactory performance in fulfillment of a designated mission is, of course, a fundamental reason for our concern with handling qualities. Why cannot the experimental measurement of performance replace pilot evaluation? Why not measure pilot-vehicle performance in the intended use - isn't good performance consonant with good quality? A significant difficulty arises here in that the performance measurement tasks may not demand of the pilot all that the real mission demands. The pilot is an adaptive controller whose goal (when so instructed) is to achieve good performance. In a specific task, he is capable of attaining essentially the same performance for a wide range of vehicle characteristics, at the expense of significant reductions in his capacity to assume other duties and planning operations. Significant differences in task performance may not be measured where very real differences in mission suitability do exist.

The questions which arise in using performance measurements may be summarized as follows: (1) For what maneuvers and tasks should measurements be made to define the mission suitability? (2) How do we integrate and weigh the performance in several tasks to give an overall measure of quality if measurable differences do exist? (3) Is it necessary to measure or evaluate pilot workload and attention factors for performance to be meaningful? If so, how are these factors weighed with those in (2)? (4) What disturbances and distractions are necessary to provide a realistic workload for the pilot during the measurement of his performance in the specified task?

Pilot evaluation still remains the only method of assessing the interactions between pilot performance and workload in determining suitability of the airplane for the mission. It is required in order to provide a basic measure of quality and to serve as a standard against which

pilot-airplane system theory may be developed, against which performance measurements may be correlated and with which significant airplane design parameters may be determined and correlated.

The technical content of the pilot evaluation generally falls into two categories: one, the identification of characteristics which interfere with the intended use, and two, the determination of the extent to which these characteristics affect mission accomplishment. The latter judgment may be formalized as a pilot rating.

In 1956, the newly formed Society of Experimental Test Pilots accepted responsibility for one program session at the annual meeting of the Institute of Aeronautical Sciences. For this purpose, a paper, entitled "Understanding and Interpreting Pilot Opinion" was prepared, which represented an attempt to create better understanding and utilization of pilot opinion and evaluation in the field of aeronautical research and development. The widespread use of rating systems has indicated a general need for some uniform method of assessing aircraft handling qualities through pilot opinion.

Several rating scales were independently developed during the early use of variable stability aircraft. These vehicles, as well as the use of ground simulation, made possible systematic studies of aircraft handling qualities through pilot evaluation and rating of the effects of specific stability and control parameters.

Figure 6.15 shows the 10-point Harper-Cooper Rating Scale that is widely used today.

CONTROLLABLE  CAPABLE OF BEING CONTROLLED OR MANAGED IN CONTEXT OF MISSION WITH AVAILABLE PILOT ATTENTION	ACCEPTABLE  MAY HAVE DEFICIENCIES WHICH WARRANT IMPROVEMENT, BUT ADEQUATE FOR MISSION.  PILOT COMPENSATION, IF REQUIRED TO ACHIEVE ACCEPTABLE PERFORMANCE, IS FEASIBLE.	SATISFACTORY  MEETS ALL REQUIREMENTS AND EXPECTATIONS, GOOD ENOUGH WITHOUT IMPROVEMENT	EXCELLENT, HIGHLY DESIRABLE	A1
			GOOD, PLEASANT, WELL BEHAVED	A2
		CLEARLY ADEQUATE FOR MISSION	FAIR. SOME MILDLY UNPLEASANT CHARACTERISTICS. GOOD ENOUGH FOR MISSION WITHOUT IMPROVEMENT	A3
		UNSATISFACTORY	SOME MINOR BUT ANNOYING DEFICIENCIES. IMPROVEMENT IS REQUESTED. EFFECT ON PERFORMANCE IS EASILY COMPENSATED FOR BY PILOT.	A4
		RELUCTANTLY ACCEPTABLE DEFICIENCIES WHICH WARRANT IMPROVEMENT, PERFORMANCE ADEQUATE FOR MISSION WITH FEASIBLE PILOT COMPENSATION.	MODERATELY OBJECTIONABLE DEFICIENCIES. IMPROVEMENT IS NEEDED. REASONABLE PERFORMANCE REQUIRES CONSIDERABLE PILOT COMPENSATION.	A5
			VERY OBJECTIONABLE DEFICIENCIES. MAJOR IMPROVEMENTS ARE NEEDED. REQUIRES BEST AVAILABLE PILOT COMPENSATION TO ACHIEVE ACCEPTABLE PERFORMANCE.	A6
	UNACCEPTABLE  DEFICIENCIES WHICH REQUIRE MANDATORY IMPROVEMENT. INADEQUATE PERFORMANCE FOR MISSION EVEN WITH MAXIMUM FEASIBLE PILOT COMPENSATION.		MAJOR DEFICIENCIES WHICH REQUIRE MANDATORY IMPROVEMENT FOR ACCEPTANCE. CONTROLLABLE. PERFORMANCE INADEQUATE FOR MISSION, OR PILOT COMPENSATION REQUIRED FOR MINIMUM ACCEPTABLE PERFORMANCE IN MISSION IS TOO HIGH.	U7
			CONTROLLABLE WITH DIFFICULTY. REQUIRES SUBSTANTIAL PILOT SKILL AND ATTENTION TO RETAIN CONTROL AND CONTINUE MISSION.	U8
			MARGINALLY CONTROLLABLE IN MISSION. REQUIRES MAXIMUM AVAILABLE PILOT SKILL AND ATTENTION TO RETAIN CONTROL.	U9
		UNCONTROLLABLE CONTROL WILL BE LOST DURING SOME PORTION OF MISSION.		UNCONTROLLABLE IN MISSION

Figure 6.15 Ten-Point Harper-Cooper Pilot Rating Scale

A flow chart is shown in figure 6.16 which traces the series of dichotomous decisions which the pilot makes in arriving at the final rating. As a rule, the first decision may be fairly obvious. Is the configuration controllable or uncontrollable? Subsequent decisions become less obvious as the final rating is approached.

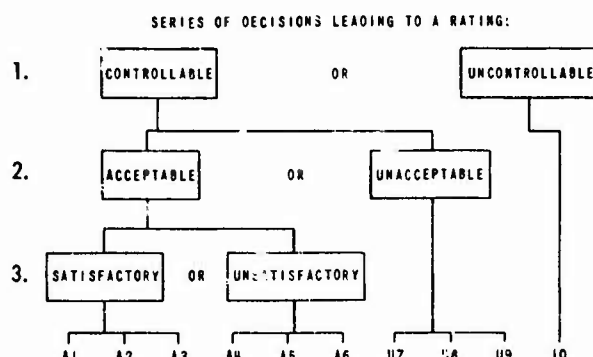


Figure 6.16 Sequential Pilot Rating Decisions

If the airplane is uncontrollable in the mission, it is rated 10. If it is controllable, the second decision examines whether it is acceptable or unacceptable. If unacceptable, the ratings U7, U8, and U9 are to be considered (rating 10 has been excluded by the "controllable" answer to the first decision). If it is acceptable, the third decision must examine whether it is satisfactory or unsatisfactory. If unsatisfactory, the ratings A4, A5 and A6 are to be considered; if satisfactory, the ratings A1, A2, and A3 are to be considered.

The basic categories must be described in carefully selected terms to clarify and standardize the boundaries desired. Following a careful review of dictionary definitions and consideration of the pilot's requirement for clear, concise descriptions, the category definitions shown in figure 6.17 were selected. When considered in conjunction with the structural outline presented in figure 6.16 a clearer picture of the series of decisions which the pilot must make is obtained.

CATEGORY	DEFINITION
CONTROLLABLE	CAPABLE OF BEING CONTROLLED OR MANAGED IN CONTEXT OF MISSION, WITH AVAILABLE PILOT ATTENTION.
UNCONTROLLABLE	CONTROL WILL BE LOST DURING SOME PORTION OF MISSION.
ACCEPTABLE	MAY HAVE DEFICIENCIES WHICH WARRANT IMPROVEMENT BUT ADEQUATE FOR MISSION. PILOT COMPENSATION, IF REQUIRED TO ACHIEVE ACCEPTABLE PERFORMANCE, IS FEASIBLE.
UNACCEPTABLE	DEFICIENCIES WHICH REQUIRE MANDATORY IMPROVEMENT. INADEQUATE PERFORMANCE FOR MISSION, EVEN WITH MAXIMUM FEASIBLE PILOT COMPENSATION.
SATISFACTORY	MEETS ALL REQUIREMENTS AND EXPECTATIONS; GOOD ENOUGH WITHOUT IMPROVEMENT. CLEARLY ADEQUATE FOR MISSION.
UNSATISFACTORY	RELUCTANTLY ACCEPTABLE. DEFICIENCIES WHICH WARRANT IMPROVEMENT. PERFORMANCE ADEQUATE FOR MISSION WITH FEASIBLE PILOT COMPENSATION.

Figure 6.17 Major Category Definitions

#### ● 6.5.2.1 MAJOR CATEGORY DEFINITIONS

Let us examine what is meant by controllable. To control is to exercise direction of, or to command. Control also means to regulate. The determination as to whether the airplane is controllable or not must be made within the framework of the defined mission or intended use. An example of the considerations of this decision would be the evaluation of fighter handling qualities during which the evaluation pilot encounters a configuration over which he can maintain control only with his complete and undivided attention. The configuration is "controllable" in the sense that the pilot can maintain control by restricting the tasks and maneuvers which he is called upon to perform, and by giving the configuration his undivided attention. However, for him to answer "Yes, it is controllable in the mission," he must be able to retain control in the mission tasks with whatever effort and attention are available from the totality of his mission duties.

The dictionary shows acceptable to mean that a thing offered is received with a consenting mind; unacceptable means that it is refused or rejected. Acceptable means that the mission can be accomplished; it means that the evaluation pilot would agree to buy it for the mission: for him to fly, for his son to fly, or for either to ride in as a passenger. "Acceptable" in the rating scale doesn't say how good it is for the mission, but it does say it is good enough. With these characteristics, the mission can be accomplished. It may be accomplished with considerable expenditure of effort and concentration on the part of the pilot, but the levels of effort and concentration required in order to achieve this acceptable performance are feasible in the intended use. By the same token, unacceptable does not necessarily mean that the mission cannot be accomplished; it does mean that the effort, concentration, and workload necessary to accomplish the mission are of such a magnitude that the evaluation pilot rejects that airplane for the mission.

Consider now a definition of satisfactory. The dictionary defines this as adequate for the purpose. A pilot's definition of satisfactory might be that it isn't necessarily perfect or even good, but it is good enough that he wouldn't ask that it be fixed. It meets a standard, it has sufficient goodness; it can meet all requirements of a mission. Acceptable but unsatisfactory implies that it is reluctantly acceptable even though objectionable characteristics should be improved, that it is deficient in a limited sense, or that there is insufficient goodness. Thus, the quality is either:

- a. Completely acceptable (satisfactory) and therefore of the best category, or
- b. Reluctantly acceptable (unsatisfactory) and of the next best category, or
- c. Unacceptable. Not suitable for the mission, but still controllable, or
- d. Unacceptable for the mission and uncontrollable.

#### ● 6.5.2.2 EXPERIMENTAL USE OF RATING OF HANDLING QUALITIES

The evaluation of handling qualities has a similarity to other scientific experiments in that the output data are only as good as the care taken in the design and execution of the experiment itself and in the analysis and reporting of the results. There are two basic categories of output data in a handling qualities evaluation: the pilot comment data and the pilot ratings. Both items are important output data. An experiment which ignores one of the two outputs is discarding a substantial part of the output information.

As one might expect, the output data which are most often neglected are the pilot comments, primarily because they are quite difficult to deal with due to their qualitative form and, perhaps, their bulk. Ratings, however, without the attendant pilot objections, are only part of the story. Only if the deficient areas can be identified, can one expect to devise improvements to eliminate or attenuate the shortcomings. The pilot comments are the means by which the identification can be made.

There are several factors which have a strong influence on the quality of pilot evaluation data and a brief discussion of them follows.

#### ● 6.5.2.3 MISSION DEFINITION

Explicit definition of the mission is probably the most important contributor to the objectivity of the pilot evaluation data. The mission is defined here as a use to which the pilot-airplane combination is to be put. The mission must be very carefully examined, and a clear definition and understanding must be reached between the engineer and the evaluation pilot as to their interpretation of this mission. This definition must include:

- a. what the pilot is required to accomplish with the airplane, and
- b. the conditions or circumstances under which he must perform the mission.

For example, the conditions or circumstances might include instrument or visual flight or both, type of displays in the cockpit, input information to assist the pilot in the accomplishment of the mission, etc. The environment in which the mission is to be accomplished must also be defined and considered in the evaluation, and could include, for example, the presence or absence of turbulence, day versus night, the frequency with which the mission has to be repeated, the variability in the preparedness of the pilot for the mission, and his level of proficiency.

#### ● 6.5.2.4 SIMULATION SITUATION

The pilot evaluation is seldom conducted under the circumstances of the real mission. The evaluation almost inherently involves simulation to some degree because of the absence of the real situation. As an example, the evaluation of a day fighter is seldom carried out under the circumstances of a combat mission in which the pilot is not only shooting at real targets, but also being shot back at by real guns. Therefore, after the mission has been defined, the relationship of the simulation

situation to the real mission must be explicitly stated for both the engineer and the evaluation pilot so that each may clearly understand the limitations of the simulation situation.

The pilot and engineer must both know what is left out of the evaluation program, and also what is in that should not be in. The fact that the anxiety and tension of the real situation are missing, and that the airplane is flying in the clear blue of calm daylight air, instead of in the icing, cloudy, turbulent, dark situation of the real mission, will affect results. Regardless of the evaluation tasks selected, the pilot must use his knowledge and experience to provide a rating which includes all considerations which are pertinent to the mission, whether provided in the tasks or not.

#### ● 6.5.2.5 PILOT COMMENT DATA

One of the fallacies resulting from the use of a rating scale which is considered for universal handling qualities application is the assumption that the numerical pilot rating can represent the entire qualitative assessment. Extreme care must be taken against this oversimplification because it does not constitute the full data gathering process.

The pilot objections to the handling qualities are important, particularly to the airplane designer who is responsible for the improvement of the handling qualities. But, even more important, the pilot comment data are essential to the engineer who is attempting to understand and use the pilot rating data. If ratings are the only output data, one has no real way of assessing whether the objectives of the experiment were actually realized. Pilot comments supply a means of assessing whether the pilot objections (which lead to his summary rating) were related to the mission or resulted from some extraneous uncontrolled factor in the execution of the experiment, or from individual pilots focusing on and weighing differently various aspects of the mission. In order that the pilot comments be most useful, several details are important.

The comments must be given by the pilot in the simplest language. Engineering terms are generally to be avoided, unless they are carefully defined. The pilot should report what he sees and feels, and describe his difficulties in carrying out that which he is attempting. It is then important for the pilot to relate the difficulties which he is having in executing specific tasks to their effect on the accomplishment of the mission.

The pilot should be required to make specific comments in evaluating each configuration. These comments generally are in response to questions which have been developed in the discussions of the mission and simulation situation. The pilot must also be free to make comments regarding his difficulties over and above the answers to the specific questions asked of him. In this regard, the test pilot should strive for a balance between a continuous running commentary and occasional comment in the form of an explicit adjective. The former often requires so much editing to find the substance that it is often ignored, while the latter may add nothing to the numerical rating itself.

The pilot comments must be taken during or immediately after each evaluation. For in-flight evaluations, this means that the comments

should be recorded on a tape recorder. Experience has shown that the best free comments are often given during the evaluation. If the comments are left until the conclusion of the evaluation, they are often forgotten. A useful procedure is to permit free comment during the evaluation itself and to require answers to specific questions in the summary comments at the end of the evaluation.

Questionnaires and supplementary pilot comments are most necessary to ensure that: (a) all important or suspected aspects are considered and not overlooked, (b) information is provided relative to why a given rating has been given, (c) an understanding is provided of the tradeoffs with which pilots must continually contend, and (d) supplementary comment that might not be offered otherwise is stimulated. It is recommended that the pilots participate in the preparation of the questionnaires. The questionnaires should be modified if necessary as a result of the pilots' initial evaluations.

#### ●6.5.2.6 PILOT RATING DATA

The pilot rating is an overall summation of the net effect of all of the objections which the pilot has observed during the evaluation as they relate to the mission. It is emphasized that the basic question that is asked of the pilot conditions the answer that he provides. For this reason, it is most important to ensure that the objectives of the program are clearly stated and understood by all concerned, and that all criteria, whether established or assumed, be clearly defined. In other words, it is extremely important that the basis upon which the evaluation is established be firmly understood by pilots and engineers. Unless a common basis is used, one cannot hope to achieve comparable pilot ratings, and confusing disagreement will often result. Care must also be taken that criteria established at the beginning of the program carry through to the end. If the pilot finds it necessary to modify his tasks, technique or mission definition during the program, he must make it clear just when this change occurred.

A discussion of the specific use of a rating scale tends to indicate some disagreement among pilots as to how they actually arrive at a specific numerical rating. There is general agreement that the numerical rating is only a shorthand for the word definition. Some pilots, however, lean heavily on the specific adjective description and look for that description which best fits their overall assessment. Other pilots prefer to make the dichotomous decisions sequentially, thereby arriving at a choice between two or three ratings. The decision among the two or three ratings is then based upon the adjective description. In concept, the latter technique is much to be preferred since it emphasizes the relationship of all decisions to the mission.

It is suggested that the actual technique used is somewhere between the two techniques above and not so different among pilots. In the past, the pilot's choice has probably been strongly influenced by the relative usefulness of the descriptions provided for the categories on one hand, and the numerical ratings on the other. The evaluation pilot is more or less continuously considering the rating decision process during his evaluation. He proceeds through the dichotomous decisions to the adjective descriptors enough times that his final decision is a blend of both techniques. It is therefore obvious that descriptors should not be contradictory to the mission-oriented framework.

Half ratings are permitted (e.g., rating 4.5) and are generally used by the evaluation pilot to indicate a reluctance to assign either of the adjacent ratings to describe the configuration. Any finer breakdown than half ratings is prohibited since any number greater than or less than the half rating implies that it belongs to the adjacent group. Any distinction between configurations assigned the same rating must be made in the pilot comments. Use of the 3.5, 6.5, and 9.5 ratings is discouraged as they must be interpreted as evidence that the pilot is unable to make the fundamental decision with respect to category.

As noted previously, the pilot rating and comments must be given on the spot in order to be most meaningful. If the pilot should later want to change his rating, the engineer should record the reasons and the new rating for consideration in the analysis, and should attempt to repeat the configuration later in the evaluation program. If the configuration cannot be repeated, the larger weight (in most circumstances) should be given to the on-the-spot rating since it was given when all the characteristics were freshest in the pilot's mind.

#### ●6.5.2.7 EXECUTION OF HANDLING QUALITIES EXPERIMENTS

Probably the most important item is the admonition to execute the experiment as it was planned. This requires careful attention to the conduct of the experiment so that the plans are actually executed in the manner intended. It is valuable for the engineer to monitor the pilot comment data as the experiment is conducted in order that he becomes aware of evaluation difficulties as soon as they occur. These difficulties may take a variety of forms. The pilot may use words which the engineer needs to have defined. The pilot's word descriptions may not convey a clear, understandable picture of the piloting difficulties. Direct communication between pilot and engineer is most important in clarifying such uncertainties. In fact, communication is probably the most important single element in the evaluation of handling qualities. Pilot and engineer must endeavor to understand one another, and cooperate to achieve and retain this understanding. The very nature of the experiment itself makes this somewhat difficult. The engineer is usually not present during the evaluation and, hence, he has only the pilot's word description of any piloting difficulty. Often, these described difficulties are contrary to the intuitive judgments of the engineer based on the characteristics of the airplane by itself. Mutual confidence is required. The engineer should be confident that the pilot will give him accurate, meaningful data; the pilot should be confident that the engineer is vitally interested in what he has to say and trusts the accuracy of his comments.

It is important that the pilot have no foreknowledge of the specific characteristics of the configuration being investigated. This does not exclude information which can be provided to help shorten certain tests (e.g., the parameter variations are lateral-directional, only). But it does exclude foreknowledge of the specific parameters under evaluation. The pilot must be free to examine the configuration without prejudice, learn all he can about it from meeting it as an unknown for the first time, look clearly and accurately at his difficulties in performing the evaluation task, and freely associate these difficulties with their effects on the ultimate success of the mission. A considerable aid to the pilot in this assessment is to present the configuration in a random-appearing fashion.

The amount of time which the pilot should use for the evaluation is difficult to specify a priori. He is normally asked to examine each configuration for as long as is necessary to feel confident that he can give a reliable and repeatable assessment. Sometimes, however, it is necessary to limit the evaluation time to a specific period of time because of circumstances beyond the control of the researcher. If the evaluation time per pilot is limited, a larger sample of pilots or repeat evaluations will be required for similar accuracy, and the pilot comment data will be of poorer quality.

One final point is the state of mind of the evaluation pilot. He must be confident of the importance of the simulation program and join wholeheartedly into the production of data which will supply answers to the questions. Pilots as a group are strongly motivated toward the production of data to improve the handling qualities of the airplanes they fly. It isn't usually necessary to explicitly motivate the pilot, but it is very important to inspire in him confidence in the structure of the experiment and the usefulness of his rating and comment data. Pilot evaluations are probably one of the most difficult tasks that a pilot undertakes. To produce useful data involves a lot of hard work, tenacity, and careful thought. There is a strong tendency for the pilot to become discouraged in the course of his evaluations about their ultimate usefulness. He worries constantly about his assessments: their accuracy and repeatability. The pilot may feel that the engineer has the answers on a sheet of paper and he is merely testing the pilot as to his ability to search out the correct answers. Such feelings are added to by a lack of communication between the piloting and engineering organizations and are to be avoided. Probably the best approach is to explicitly state to the pilot that only he knows the answers to the questions which are being asked, and he can arrive at these correct answers by carrying out the evaluation program. He must be reassured in the course of the program that his assessments are good, so that he gains confidence in the manner in which he is carrying out the program.

## ■ 6.6 CONTROL INPUTS

There are several different control inputs that could be used to excite the dynamic modes of motion of an aircraft. To accomplish the task of obtaining the free response of an aircraft, the pilot makes an appropriate control input, removes himself from the loop, and observes the resulting aircraft motion. Three inputs that are frequently used in stability and control investigations will be discussed in this section: the step input, the pulse, and the doublet.

### ● 6.6.1 STEP INPUT

When a step input is made, the applicable control is rapidly moved to a desired new position and steadily held there. The aircraft motion resulting from this suddenly applied new control position can then be recorded for analysis. A mathematical representation of a step input assumes the deflection occurs in zero time and is contrasted to a typical actual control position time history in figure 6.18. The "unit step" input is frequently used in theoretical analysis and has the magnitude of one radian, which is equivalent to 57.3 degrees. Specifying control inputs in dimensionless radians instead of degrees is convenient for use in the non-dimensional equations of motion.

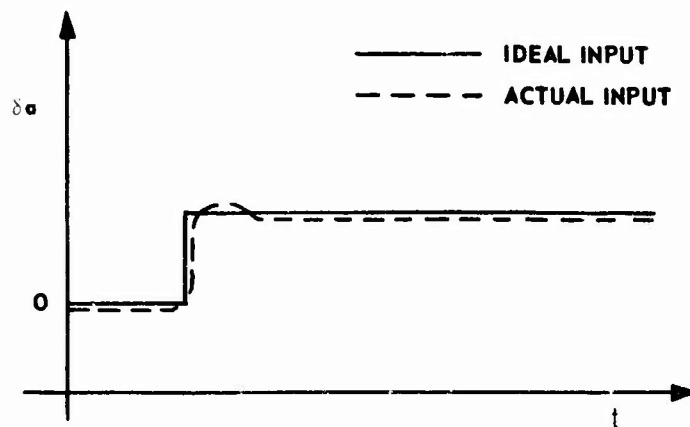


Figure 6.18 Step Input

#### ● 6.6.2 PULSE

When a pulse, or singlet, input is applied, the control is moved rapidly to a desired position, held momentarily, and then rapidly returned to its original position. The pilot can then remove himself from the loop and observe the free aircraft response. Again, deflections are theoretically assumed to occur instantaneously, and an example of a pulse, or singlet is shown in figure 6.19.

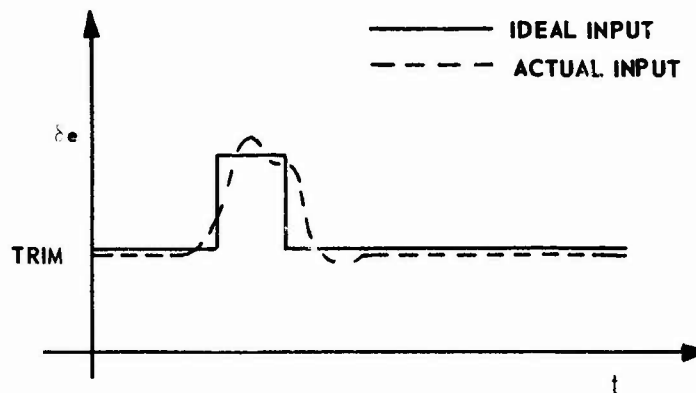


Figure 6.19 Pulse Input

The "unit impulse" input is frequently used in theoretical analysis and is related to the pulse input. The unit impulse is the mathematical result of a limiting process which begins with a pulse having an area of unity under the rectangle formed by the input and ends with an infinitely large magnitude input applied in zero time.

### 6.6.3 DOUBLET

A doublet input is a double pulse which is skew symmetric with time. After exciting a dynamic mode of motion with this input and removing himself from the control loop, the pilot can record the aircraft open loop motion. Figure 6.20 depicts a theoretical doublet input.

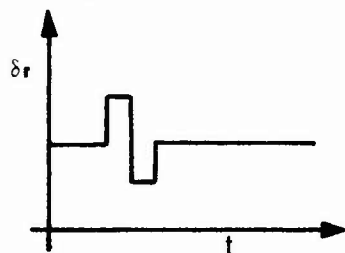


Figure 6.20 Doublet Input

### 6.7 EQUATIONS OF MOTION

Six equations of motion (three translational and three rotational) for a rigid body flight vehicle are required to solve its motion problem. A rigid body aircraft and constant mass were assumed, and the equations of motion were derived and expressed in terms of a coordinate system fixed in the body. Solving for the motion of a rigid body in terms of a body fixed coordinate system is particularly convenient in the case of an aircraft when the applied forces are most easily specified in the body axis system.

"Stability axes" were used to specify the body fixed coordinate system. With the vehicle at reference flight conditions the  $x$  axis is aligned into the relative wind; the  $z$  axis is 90 degrees from the  $x$  axis in the aircraft plane of symmetry, with positive direction down relative to the vehicle; the  $y$  axis completes the orthogonal triad. This  $xyz$  coordinate system is then fixed in the vehicle and rotates with it when perturbed from the reference equilibrium conditions. The solid lines in figure 6.21 depict initial alignment of the stability axes, and the dashed lines show the perturbed coordinate system.

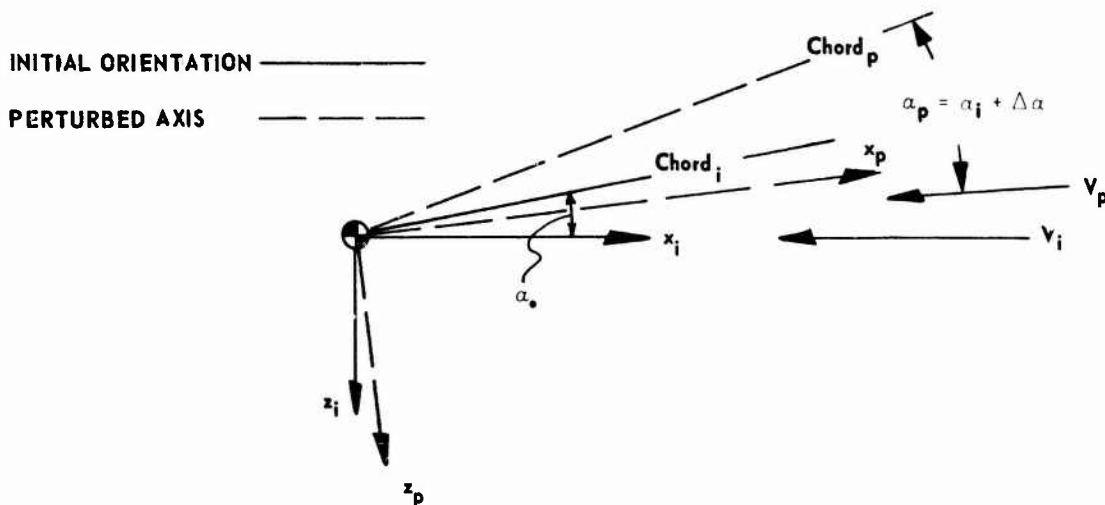


Figure 6.21 Stability Axis System

Chapter 1 contains the derivation of the complete equations of motion, and the results are listed here for your convenience.

$$\begin{aligned}
 F_x &= m (\dot{u} + qw - rv) \\
 F_y &= m (\dot{v} + ru - pw) \\
 F_z &= m (\dot{w} + pv - qu) \\
 L &= \dot{p} I_x + qr (I_z - I_y) - (\dot{r} + pq) I_{xz} \\
 M &= \dot{q} I_y - pr (I_z - I_x) + (p^2 - r^2) I_{xz} \\
 N &= \dot{r} I_z + pq (I_y - I_x) + (qr - \dot{p}) I_{xz}
 \end{aligned}
 \tag{6.19}$$

where  $F_x$ ,  $F_y$ , and  $F_z$  are forces in the  $x$ ,  $y$ , and  $z$  direction, and  $L$ ,  $M$ , and  $N$  are moments about the  $x$ ,  $y$ , and  $z$  axes taken at the vehicle center of mass.

#### 6.7.1 SEPARATION OF THE EQUATIONS OF MOTION

When all lateral-directional forces, moments, and accelerations are constrained to be zero, the equations which govern pure longitudinal motion result from the six general equations of motion. That is, substituting

$$\begin{aligned}
 p &= 0 = r \\
 \dot{p} &= 0 = \dot{r} \\
 L &= 0 = N \\
 F_y &= 0 \\
 v &= 0 \\
 \dot{v} &= 0
 \end{aligned}
 \tag{6.20}$$

into the equations labeled 6.19 results in the longitudinal equations of motion.

$$\begin{aligned}
 F_x &= m (\dot{u} + qw) \\
 F_z &= m (\dot{w} - qu) \\
 M &= \dot{q} I_y
 \end{aligned}
 \tag{6.21}$$

Linearization of the equations labeled 6.21 by Taylor series first order approximations of the forcing functions and small perturbation assumptions for the variables  $u$ ,  $q$ , and  $w$  result in a set of workable equations for longitudinal motion. Note that the resulting equations are the longitudinal perturbation equations, and that the unknowns are the perturbed values of  $\alpha$ ,  $u$ , and  $\theta$  from the equilibrium condition. These equations in coefficient form are<sup>2</sup>

<sup>2</sup>Automatic Control of Aircraft and Missiles, Blakelock, Wylie, 1965.

$$\left. \begin{aligned} \left( \frac{mU_o}{Sq} \dot{u} - C_{x_u} u \right) + \left( \frac{-c}{2U_o} C_{x_\alpha} \dot{\alpha} - C_{x_\alpha} \alpha \right) + \left( \frac{-c}{2U_o} C_{x_q} \dot{\theta} + \frac{mg}{Sq} \theta \right) &= C_{x_{\delta e}} \delta e \\ \left( -C_{z_u} u \right) + \left[ \left( \frac{mU_o}{Sq} - \frac{c}{2U_o} C_{z_\alpha} \right) \dot{\alpha} - C_{z_\alpha} \alpha \right] + \left[ \left( \frac{-mU_o}{Sq} - \frac{c}{2U_o} C_{z_q} \right) \dot{\theta} \right] &= C_{z_{\delta e}} \delta e \\ \left( -C_{M_u} u \right) + \left( \frac{-c}{2U_o} C_{M_\alpha} \dot{\alpha} - C_{M_\alpha} \alpha \right) + \left[ \frac{I_y}{Sq c} \ddot{\theta} - \frac{c}{2U_o} C_{M_q} \dot{\theta} \right] &= C_{M_{\delta e}} \delta e \end{aligned} \right\} (6.22)$$

In the equations labeled 6.22,

$q = \frac{1}{2} \rho U_o^2$  (except when  $q$  is a subscript denoting a partial derivative with respect to pitch rate.)

$u = \frac{\Delta U}{U_o}$  (A dimensionless velocity parameter has been defined for convenience.)

$\alpha, \theta$  are perturbations about their equilibrium values.

$C_{x_u}, C_{z_\alpha}$ , etc., are partial derivatives evaluated at the reference conditions with respect to force coefficients. Derivations for these terms may be found in Blakelock.

Note that the equations labeled 6.22 are for pure longitudinal motion and that the unknowns are perturbation values about the reference conditions.

Laplace transforms can be used to facilitate solutions to the longitudinal perturbation equations. For example, taking Laplace transforms of the  $x$  force equation and stating that initial perturbation values are zero results in

$$\left[ \frac{mU_o}{Sq} s - C_{x_u} \right] u(s) + \left[ \frac{-c}{2U_o} C_{x_\alpha} s - C_{x_\alpha} \right] \alpha(s) + \left[ \frac{-c}{2U_o} C_{x_q} s + \frac{mg}{Sq} \right] \theta(s) = C_{x_{\delta e}} \delta e(s) \quad (6.23)$$

The other two equations could similarly be Laplace transformed to obtain a set of longitudinal perturbation equations in the  $S$  domain.

## 6.6 LONGITUDINAL MOTION

The equations labeled 6.22 describe the perturbed longitudinal motion of an aircraft about some equilibrium conditions. The theoretical solutions for aircraft motion can be quite good, depending on the accuracy of the various aerodynamic parameters. For example,

$$C_{x_u} = C_L - C_{D_u}$$

is one parameter appearing in the  $x$  force equation, and the goodness of the solution will certainly depend on how accurately the values of  $C_L$  and  $C_{D_0}$  are known. Before an aircraft flies, such values result from theoretical predictions and wind tunnel data. After appropriate flight tests have been flown, values for the various stability derivatives can be extracted from flight test data.

### 6.8.1 EXAMPLE PROBLEM

Blakelock presents an example problem for a four-engine jet transport using the longitudinal equations to solve for the perturbed aircraft motion. The reference flight conditions are straight and level at 40,000 feet with a velocity of 600 feet per second. Values for the various aerodynamic parameters are specified, and the set of longitudinal equations in the Laplace domain become

$$\begin{aligned} [13.78S + .088]u(S) - .392\alpha(S) + .74\theta(S) &= C_{x_{\delta e}} \delta e(S) \\ 1.48u(S) + [13.78S + 4.46]\alpha(S) - 13.78S\theta(S) &= C_{z_{\delta e}} \delta e(S) \\ 0 + [.0552S + .619]\alpha(S) + [.514S^2 + .192S]\theta(S) &= C_{M_{\delta e}} \delta e \end{aligned} \quad (6.24)$$

These equations are of the form

$$\left. \begin{aligned} au + b\alpha + c\theta &= d \\ eu + f\alpha + g\theta &= h \\ iu + j\alpha + k\theta &= l \end{aligned} \right\} \quad (6.25)$$

and the set of equations can be readily solved for any of the variables. For example, from equation 6.25,

$$u(S) = \frac{\begin{vmatrix} a & d & c \\ e & h & g \\ i & l & k \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}} = \frac{\text{Numerator}(S)}{\text{Denominator}(S)}$$

Recall that the denominator of the above equation in the  $S$  domain is the system characteristic equation and that the location of the roots of  $\Delta(S)$  will immediately indicate the type of dynamic response. From equation 6.24

$$\Delta(S) = 97.5S^4 + 79S^3 + 128.9S^2 + .998S + .677 \quad (6.26)$$

Factoring higher order equations such as 6.26 is not a simple thing, but some systematic approaches do exist. Blakelock refers to Lin's method and accomplishes the factoring of 6.26 into two quadratics.

$$\Delta(S) = (S^2 + .00466S + .005)(S^2 + .806S + 1.211) \quad (6.27)$$

Each of the quadratics listed in equation 6.27 will have a natural frequency and damping ratio associated with it, and the values can be rapidly computed by comparing the particular quadratic to the standard notation second order characteristic equation:

$$\begin{aligned}\zeta_1 &= 0.352 \\ \omega_{n_1} &= 1.145 \text{ radians/sec}\end{aligned}\tag{6.28}$$

$$\begin{aligned}\zeta_2 &= 0.032 \\ \omega_{n_2} &= 0.073 \text{ radians/sec}\end{aligned}\tag{6.29}$$

### 6.8.2 LONGITUDINAL MOTION MODES

Experience has shown that aircraft exhibit two different types of longitudinal oscillations:

1. One of short period with relatively heavy damping that is called the "short period" mode.
2. Another of long period with very light damping that is called the "phugoid" mode.

The periods and damping of these oscillations vary from aircraft to aircraft and with flight conditions.

The short period mode is characterized primarily by variations in angle of attack and pitch angle with very little change in forward speed. Relative to the phugoid, the short period has a high frequency and heavy damping.

Typical values for its damped period are in the range of 2 to 5 seconds. Generally, the short period motion is the more important longitudinal mode for handling qualities since it is contributing to the motion being observed and corrected by the pilot when the pilot is in the loop.

The phugoid mode is characterized mainly by variations in  $u$  and  $\theta$  with  $\alpha$  nearly constant. This long period oscillation can be thought of as a constant total energy problem with exchanges between potential and kinetic energy. The aircraft nose drops and airspeed increases as the aircraft descends below its initial altitude. Then the nose rotates up, causing the aircraft to climb above its initial altitude with airspeed decreasing until the nose lazily drops below the horizon at the top of the maneuver.

Because of light damping, many cycles are required for this motion to damp out. However, its long period combined with low damping results in an oscillation that is easily controlled by the pilot, even for a slightly divergent motion. When the pilot is in the loop, he is frequently not aware that the phugoid mode exists as he makes control inputs and obtains aircraft response before the phugoid can be seen. Typical values for its damped period range in the order of 45 to 90 seconds.

From the example problem in section 6.8.1 and from the descriptions of the longitudinal mode, it is possible to immediately specify which parameters are associated with the short period and which must be that aircraft's phugoid parameters.

### ● 6.8.3 SHORT PERIOD APPROXIMATION EQUATIONS

A logical approach to use when trying to get a simplified set of equations to describe the short period mode is to recall that the short period occurs at nearly constant airspeed and set  $u = 0$  in the equations labeled 6.22. The result of this substitution is two unknowns appearing in three independent equations, and it is certainly desirable to select the correct set of two equations for solution. Note that the  $x$  force equation could be expected to contribute primarily to a change of velocity in the  $x$  direction; however the specification that  $u = 0$  has been made for this approximation. Choosing to discard the  $x$  force equation and making the above substitutions results in a set of two equations with the unknowns  $\alpha$  and  $\theta$ . In the  $S$  domain these equations are

$$\left. \begin{aligned} \left[ \frac{mU_o}{Sq} S - C_{z_\alpha} \right] \alpha(S) + \left[ \frac{-mU_o}{Sq} S \right] \theta(S) &= C_{z_{\delta e}} \delta e(S) \\ \left[ -\frac{c}{2U_o} C_{M_\alpha} S - C_{M_\alpha} \right] \alpha(S) + \left[ \frac{I_y}{Sq c} S^2 - \frac{c}{2U_o} C_{M_q} S \right] \theta(S) &= C_{M_{\delta e}} \delta e(S) \end{aligned} \right\} \quad (6.29)$$

where  $C_{z_\alpha}$  and  $C_{z_q}$  have been assumed to be negligible.

As for previous examples, recall that the characteristic equation can be found by expanding the determinant of the coefficients from the left side of the equations labeled 6.29. From these equations, the characteristic equation is

$$\Delta(S) = S (AS^2 + BS + C) \quad (6.30)$$

where

$$\begin{aligned} A &= \left( \frac{I_y}{Sq c} \right) \left( \frac{mU_o}{Sq} \right) \\ B &= \left( \frac{-c}{2U_o} C_{M_q} \right) \left( \frac{mU_o}{Sq} \right) - \left( I_y C_{z_\alpha} \right) - \left( \frac{c}{2U_o} C_{M_\alpha} \right) \left( \frac{mU_o}{Sq} \right) \\ C &= \left( \frac{c}{2U_o} C_{M_q} C_{z_\alpha} \right) - \left( \frac{mU_o}{Sq} C_{M_\alpha} \right) \end{aligned}$$

To find expressions for the short period damping ratio and natural frequency from the second order part of equation 6.30, it can be rewritten as

$$\Delta(S) = S \left( S^2 + \frac{B}{A} S + \frac{C}{A} \right)$$

and compared to the standard notation second order characteristic equation.

Some rather involved expressions result for  $\zeta$  and  $\omega_n$

$$\zeta = -\frac{1}{4} \left( C_{M_q} + C_{M_\alpha} + \frac{2I_y}{mc^2} C_{Z_\alpha} \right) \left[ \frac{mc^2}{I_y \frac{C_{M_q} C_{Z_\alpha}}{2} - \frac{2m C_{M_\alpha}}{\rho S c}} \right]^{1/2}$$

$$\omega_n = \frac{U_o \rho S c}{2} \left[ \frac{\frac{C_{M_q} C_{Z_\alpha}}{2} - \frac{2m}{\rho S c} C_{M_\alpha}}{I_y m} \right]^{1/2} \quad (6.31)$$

If the aircraft parameter values that were used in the example problem from section 6.8.1 are substituted in the equations labeled 6.31, the following values result for  $\zeta$  and  $\omega_n$

$$\zeta = .35 \quad \omega_n = 1.15 \quad (6.32)$$

Comparison of the above to the values shown in the equations labeled 6.28 show good agreement for this problem.

The complicated expressions listed in the equations labeled 6.31 can be further simplified by discarding the terms that are usually the smallest contributors to the expressions. Stating that the  $C_{M_\alpha}$  and  $C_{Z_\alpha}$  terms in the numerator of the  $\zeta$  expression are negligible when compared to  $C_{M_q}$  and that the  $C_{M_q} C_{Z_\alpha}$  term in the denominator is small compared to the  $C_{M_\alpha}$  term results in a more simplified expression for  $\zeta$ . This functional relationship can be used to predict trends in the short period damping ratio as flight conditions are changed.

$$\zeta \approx \zeta \left[ \left( \sqrt{\frac{c}{I_y}} \right) \left( \frac{-C_{M_q}}{\sqrt{-C_{M_\alpha}}} \right) \right] \quad (6.33)$$

Stating that the  $C_{M_\alpha}$  term is the significant one in the numerator of the  $\omega_n$  expression listed in 6.31 results in

$$\omega_n \approx \omega_n \left[ U_o \left( \sqrt{\frac{\rho}{I_y}} \right) \left( \sqrt{\frac{S c}{2}} \right) \left( \sqrt{-C_{M_\alpha}} \right) \right] \quad (6.34)$$

Equation 6.34 can be used to predict the trends expected in the short period natural frequency as flight conditions change. Both equations 6.33 and 6.34 show the predominant stability derivatives which affect the short period damping ratio and natural frequency.

#### 6.8.4 PHUGOID APPROXIMATION EQUATIONS

An approach similar to that used when obtaining the short period approximation will be used to obtain a set of equations to approximate the phugoid oscillation. Recalling that the phugoid motion occurs at

nearly constant angle of attack, it is logical to substitute  $\alpha = 0$  into the longitudinal motion equations. This results in a set of three equations with only two unknowns. Further reasoning that the phugoid motion is characterized primarily by altitude excursions and changes in aircraft speed implies that the  $z$  force and  $x$  force equations are the two equations which should be used. The resulting set of two equations for the phugoid approximation in the Laplace domain is

$$\begin{aligned} \left[ \frac{mU_0}{Sq} s - C_{x_u} \right] u(s) + \left[ \frac{mg}{Sq} \right] \theta(s) &= C_{x_{\delta e}} \delta e(s) \\ \left[ -C_{z_u} \right] u(s) + \left[ \frac{-mU_0}{Sq} s \right] \theta(s) &= C_{z_{\delta e}} \delta e(s) \end{aligned}$$

where  $C_{x_q}$  and  $C_{z_q}$  have been assumed to be negligibly small.

The characteristic equation for the phugoid approximation can now be found using the above equations.

$$\Delta(s) = \left[ \frac{-mU_0}{Sq} \right]^2 s^2 + \left[ \frac{mU_0}{Sq} C_{x_u} \right] s + \left[ \frac{mg}{Sq} \right] C_{z_u} \quad (6.35)$$

Note that lift and weight are not equal during phugoid motion, but also realize that the net difference between lift and weight is quite small. If the approximation is made that

$$L = W$$

and then the substitution that

$$W = mg$$

it can be written that

$$\frac{mg}{Sq} = C_L$$

The phugoid characteristic equation can thus be rewritten as

$$\Delta(s) = s^2 - \frac{C_{x_u}}{\frac{mU_0}{Sq}} s - \frac{C_L}{\left( \frac{mU_0}{Sq} \right)^2} C_{z_u}$$

The phugoid natural frequency is then found to be

$$\omega_n = \frac{U_0}{2m} \sqrt{-C_{z_u} C_L} \quad (6.36)$$

In an effort to further simplify equation 6.36, the  $C_{z_u}$  terms should be examined. Examining figure 6.22

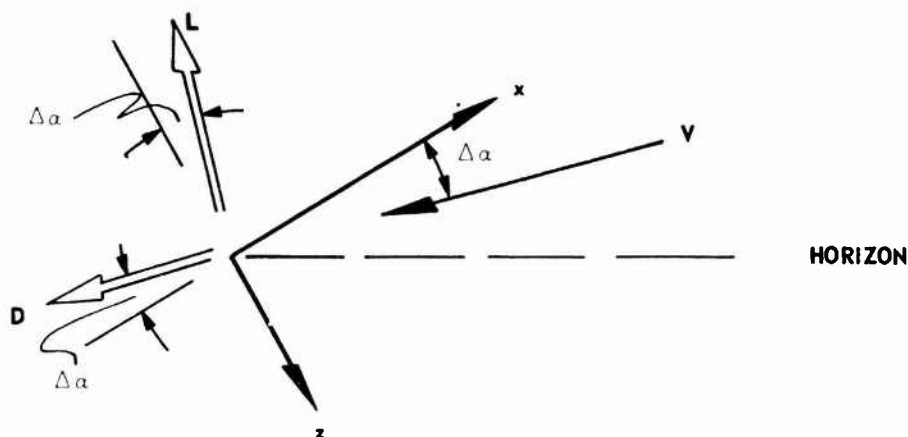


Figure 6.22

leads to

$$F_z = -L \cos \alpha - D \sin \alpha$$

then,

$$\frac{\partial F_z}{\partial u} = -\frac{\partial L}{\partial u} \cos \alpha + L \sin \alpha \frac{\partial \alpha}{\partial u} - \frac{\partial D}{\partial u} \sin \alpha - D \cos \alpha \frac{\partial \alpha}{\partial u}$$

which must be evaluated at the equilibrium conditions for use in the first order Taylor Series that was used in obtaining the linearized equations of motion

$$\left. \frac{\partial F_z}{\partial u} \right|_0 = -\frac{\partial L}{\partial u}$$

And,

$$\frac{\partial F_z}{\partial u} = -\frac{\partial}{\partial u} C_L S \frac{1}{2} \rho V^2$$

When the approximation that  $V = U$  is made in the above equation, the result is

$$\frac{\partial F_z}{\partial u} = -\frac{\partial C_L}{\partial u} S \frac{1}{2} \rho U^2 - C_L S \rho U$$

In order to have a nondimensional expression, the correct factor must be used

$$C_{z_u} = \frac{U_0}{S q} \frac{\partial F_z}{\partial u}$$

So that

$$C_{z_u} = -U_o C_{L_u} - 2 C_L \quad (6.37)$$

The variation of  $C_L$  with velocity is primarily due to Mach effects, and except for the transonic regime,  $C_{L_u}$ , is approximately zero in many cases. Figure 6.23 shows a typical plot.

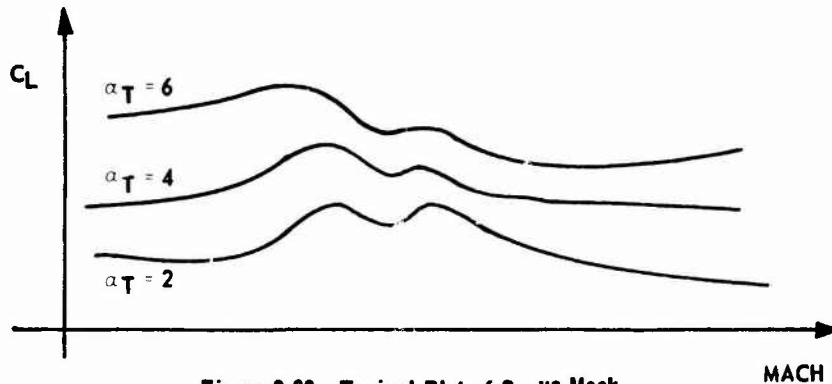


Figure 6.23 Typical Plot of  $C_L$  vs Mach

If the aircraft and flight conditions are such that  $C_{L_u} \approx 0$ , then

$$C_{z_u} = -2 C_L$$

Substituting the above equation into the expression for the phugoid natural frequency results in

$$\omega_n = \frac{Sq}{mU_o} \sqrt{2} C_L$$

An equation showing that the phugoid natural frequency is inversely related to aircraft velocity results from substituting  $C_L = \frac{mg}{Sq}$  into the above equation.

$$\omega_n = \frac{45.5}{U_o} \quad (6.38)$$

Where  $U_o$  is true velocity in feet per second.

A simplified approximate expression for the phugoid damping ratio can also be obtained and is given by

$$\zeta = \left( \frac{1}{\sqrt{2}} \right) \left( \frac{C_D}{C_L} \right) \quad (6.39)$$

Equations 6.38 and 6.39 can be used to understand some major contributors to the natural frequency and damping ratio of the phugoid motion.

#### ● 6.8.5 EQUATION FOR $n/\alpha$

Noting that the requirements of MIL-F-8785 for the short period natural frequency are stated as a function of  $n/\alpha$ , it is desirable to develop a theoretical capability to predict  $n/\alpha$ . Consider the  $z$  force equation for longitudinal motion.

$$\Sigma \Delta F_z = m (\dot{w} - U_0 \dot{\theta})$$

and recall that Newton's Second Law is a directional relationship

$$\bar{F} = m \bar{a}$$

Thus

$$a_z = \dot{w} - U_0 \dot{\theta} \quad (6.40)$$

where all the variables appearing in equation 6.40 are perturbations about the equilibrium condition. Rewriting the above equation and stating that  $a_z \approx n$

$$n = U_0 \left( \frac{\dot{w}}{U_0} - \dot{\theta} \right)$$

$$n = U_0 (\dot{\alpha} - \dot{\theta}) \quad (6.41)$$

Of course expressions for  $\dot{\alpha}$  and  $\dot{\theta}$  can be obtained from the short period solutions for  $\alpha(t)$  and  $\theta(t)$ , and an expression that gives  $n(t)$  can be written from equation 6.41. The ratio of the magnitudes of the  $n$  and  $\alpha$  envelopes can then be used to determine  $n/\alpha$ .

Using a theoretically obtained  $n/\alpha$  along with the short period natural frequency and damping ratio obtained from the equations of motion makes it possible to accomplish a design problem to check whether or not the aircraft is being designed to comply with the MIL-F-8785.

### ■ 6.9 LATERAL-DIRECTIONAL MOTION MODES

There are three typical asymmetric modes of motion exhibited by aircraft. These modes are the roll, spiral, and Dutch roll.

#### ● 6.9.1 ROLL MODE

The roll mode is considered to be a first order response which describes the aircraft roll rate response to an aileron input. Figure 6.24 depicts an idealized roll rate time history to a step aileron input. The roll mode time constant is normally small, with a MIL-F-8785 requirement to be less than three seconds.

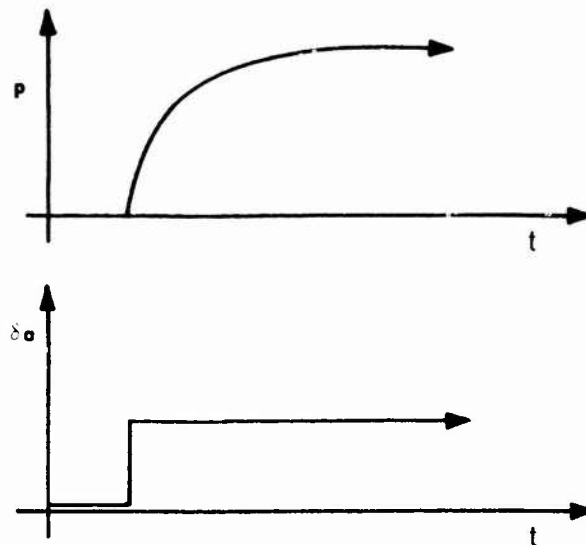


Figure 6.24 Typical Roll Mode

#### ●6.9.2 SPIRAL MODE

The spiral mode is considered to be a first order response which describes the aircraft bank angle time history as  $\phi$  tends to increase or decrease from a small, non-zero bank angle. After a wings level trim shot, the spiral mode can be observed by releasing the aircraft from bank angles as great as 20 degrees and allowing the spiral mode to occur without control inputs. If this mode is divergent, the aircraft nose continues to drop as the bank angle continues to increase, resulting in the name, "spiral mode." This mode, similar to the phugoid in that a pilot can easily control it even if it is dynamically unstable, has somewhat loose requirements in MIL-F-8785. A typical divergent time history as shown in figure 6.25 and might be characterized by  $T_2$ , the time to double amplitude.

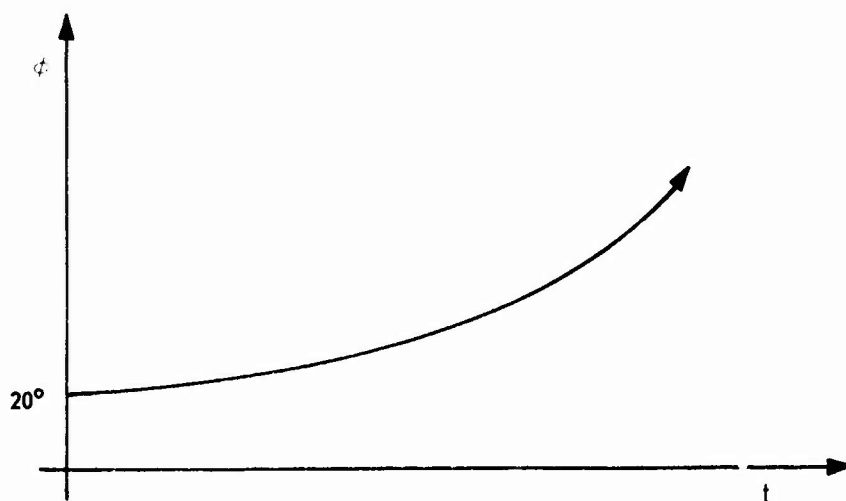


Figure 6.25 Typical Spiral Mode

### ● 6.9.3 DUTCH ROLL MODE

The Dutch roll mode is a tightly coupled yawing and rolling motion with a relatively high frequency. Some typical values for Dutch roll damped period at a cruise condition are 3 seconds for the A-7, and 3 seconds for the B-58. Typically, as the aircraft nose yaws to the right a right roll due to the yawing motion is generated. The combination of restoring forces and moments, damping, and aircraft inertia is generally such that after the motion peaks out to the right, a nose left yawing motion begins accompanied by a roll to the left. This coupled right - left - right - . . . motion often is lightly damped with a relatively high frequency.

One of the pertinent Dutch roll parameters is  $\phi/\beta$ , the ratio of bank angle to yaw angle. A very low value for  $\phi/\beta$  implies little bank action during the Dutch roll. In the limit when  $\phi/\beta$  is zero, the Dutch roll motion consists of a pure yawing motion that most pilots consider less objectionable than a Dutch roll mode with a high value for  $\phi/\beta$ .

Another parameter that can be used to characterize the Dutch roll or any other second order motion is the number of cycles required to damp to half amplitude,  $C_{1/2}$ .

A doublet rudder input is frequently used to excite the Dutch roll, and figure 6.26 shows a typical Dutch roll time history.

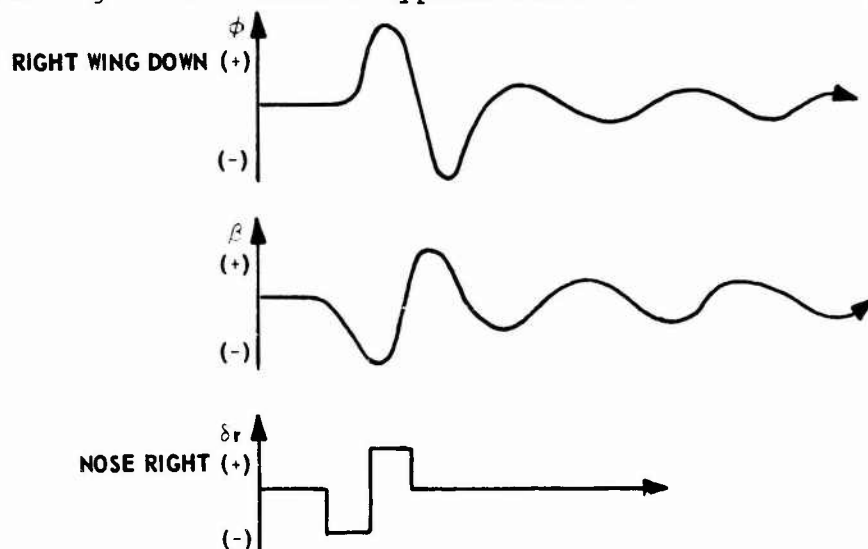


Figure 5.26 Typical Dutch Roll Mode

### ● 6.9.4 ASYMMETRIC EQUATIONS OF MOTION

Similarly to the separation of the longitudinal equations, the set of equations which describe lateral-directional motion can be separated from the six general equations of motion. Starting with equilibrium conditions and specifying that only asymmetric forcing functions, velocities, and accelerations exist results in the lateral-directional equations of motion. Assuming small perturbations and using a linear Taylor Series

approximation for the forcing functions result in the linear, lateral-directional perturbation equations of motion

$$\left. \begin{aligned} \frac{-b}{2U_o} C_{Y_p} \dot{\phi} - C_{Y_\delta} + \left( \frac{mU_o}{Sq} - 2U_o C_{Y_r} \right) \dot{\psi} - C_{Y_\psi} \psi + \frac{mU_o}{Sq} \dot{\beta} - C_{Y_\beta} \beta &= C_{Y_{\delta r}} \delta r + C_{Y_{\delta a}} \delta a \\ \frac{I_x}{Sq b} \ddot{\phi} - \frac{b}{2U_o} C_{l_p} \dot{\phi} + \frac{I_{xz}}{Sq b} \ddot{\psi} - \frac{b}{2U_o} C_{l_r} \dot{\psi} - C_{l_\beta} \beta &= C_{l_{\delta r}} \delta r + C_{l_{\delta a}} \delta a \\ \frac{I_{xz}}{Sq b} \ddot{\phi} - \frac{b}{2U_o} C_{N_p} \dot{\phi} + \frac{I_z}{Sq b} \ddot{\psi} - \frac{b}{2U_o} C_{N_r} \dot{\psi} - C_{N_\beta} \beta &= C_{N_{\delta r}} \delta r + C_{N_{\delta a}} \delta a \end{aligned} \right\} (6.42)$$

Note that the lateral-directional equations of motion have been non-dimensionalized by span,  $b$ , as opposed to chord. Also, recall that the stability derivative  $C_{l_p}$  is not a lift-referenced stability derivative but that the script  $l$  refers to rolling moment.

It is appropriate to point out that if the products of perturbation are not small, then the lateral-directional motion will couple directly into longitudinal motion. This can be readily seen by examination of the pitching moment equation and makes the point that asymmetric motion can couple into symmetric motion. Our analysis will assume that conditions are such that coupling does not exist.

#### 6.9.5 ROOTS OF $\Delta(s)$ FOR ASYMMETRIC MOTION

Laplace transforming the equations labeled 6.42 puts them into a form that readily yields the characteristic equation for asymmetric motion or that can be used to find the time response for some specified input.

The roots of the lateral directional characteristic equation typically are comprised of a relatively large negative real root, a small real root that is either positive or negative, and a complex conjugate pair of roots.

The large real root is the one associated with the roll mode of motion. Note that a large negative value for this root implies a fast time constant.

The small real root that might be either positive or negative is associated with the spiral mode. A slowly changing time response results from this small root, and the motion is either stable for a negative root or divergent for a positive root.

The complex conjugate pair of roots corresponds to the Dutch roll mode which frequently exhibits high frequency and light damping for SAS off conditions. This second order motion is of great interest in handling qualities investigations.

Figure 6.24 shows typical characteristic equation root locations for the asymmetric motion modes.

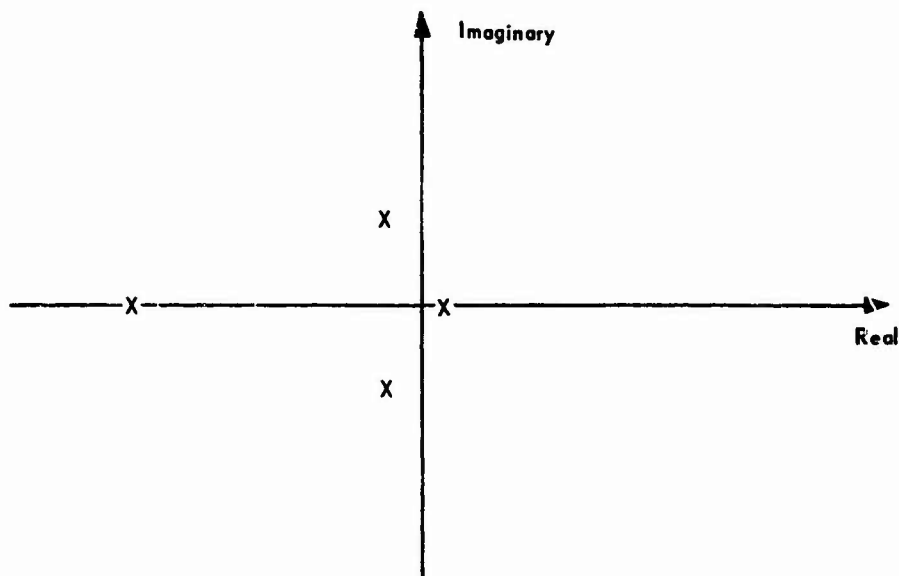


Figure 6.27 Typical Roots of  $\Delta(S)$  for Asymmetric Motion

#### ● 6.9.6 APPROXIMATE ROLL MODE EQUATION

This approximation results from the hypothesis that only rolling motion exists and use of the rolling moment equation as done in Blakelock (reference 2). The roll mode approximation equation is

$$\left[ \frac{I_{xx}}{Sq b} s - \frac{b}{2U_o} C_{\ell p} \right] \dot{\phi}(s) = C_{\ell \delta a} \delta a \quad (6.43)$$

The roll mode characteristic equation root is

$$s_R = \frac{b^2 s_{\rho} U_o}{4 I_{xx}} C_{\ell p} \quad (6.44)$$

Note that  $C_{\ell p}$  less than zero implies stability for the roll mode and that a larger negative value of  $s_R$  implies an aircraft that approaches its steady state roll rate quickly. A functional analysis can be made using equation 6.44 to predict change trends in  $\tau_R$ , the roll mode time constant, as flight conditions change.

#### ● 6.9.7 SPIRAL MODE STABILITY

Blakelock lists the condition for dynamic spiral stability, namely that

$$C_{\ell r} C_{N_r} > C_{N_{\dot{\phi}}} C_{y_r}$$

and points out that increasing  $C_{l_\beta}$  while decreasing  $C_{l_r}$  is a reasonable design method if increasing spiral stability is desired. Also he lists an equation to calculate the spiral mode  $\Delta(S)$  root

$$S_s = \frac{Sg}{mU_o} C_{Y_\psi} \left[ \frac{C_{l_\beta} C_{N_n} - C_{N_\beta} C_{l_r}}{C_{l_p} C_{N_\beta}} \right] \quad (6.45)$$

#### ● 6.9.8 DUTCH ROLL MODE APPROXIMATE EQUATIONS

The approximate equations for Dutch roll motion can be obtained by using the equations labeled 6.42 and specifying that pure sideslip exists ( $\beta = -\psi$ ) and bank angle is zero. While this specification is generally not true, the result is a reasonable approximation for the Dutch roll damping ratio and natural frequency:

$$\left. \begin{aligned} \zeta &= \left( \frac{1}{8} \sqrt{2Sb^3} \right) \left( -C_{N_r} \right) \left( \frac{\rho}{I_z C_{N_\beta}} \right)^{\frac{1}{2}} \\ \omega_n &= \sqrt{\frac{Sb}{2}} U_o \left( \frac{C_{N_\beta} \rho}{I_z} \right)^{\frac{1}{2}} \end{aligned} \right\} \quad (6.46)$$

An approximate functional relationship can be found for the magnitude of  $\phi$  to  $\beta$

$$\left| \frac{\phi}{\beta} \right| = \frac{\phi}{\beta} \left[ \left( \frac{C_{l_\beta}}{C_{N_\beta}} \right) \left( \frac{I_z}{I_x} \right) \left( \frac{1}{\rho U_o} \right) \right] \quad (6.47)$$

Equation 6.47 is of value in predicting trends in  $\phi/\beta$  as flight conditions are changed.

#### ● 6.9.9 COUPLED ROLL-SPIRAL MODE

This mode of lateral-directional motion has rarely been exhibited by aircraft, but the possibility exists that it can indeed happen. If this mode is present, the characteristic equation for asymmetric motion has two pairs of complex conjugate roots instead of the usual one complex conjugate pair along with two real roots. The phenomenon which occurs is the roll mode root decreases in absolute magnitude while the spiral mode root becomes more negative until they meet and split off the real axis to form a second complex conjugate pair of roots, as depicted in figure 6.28.

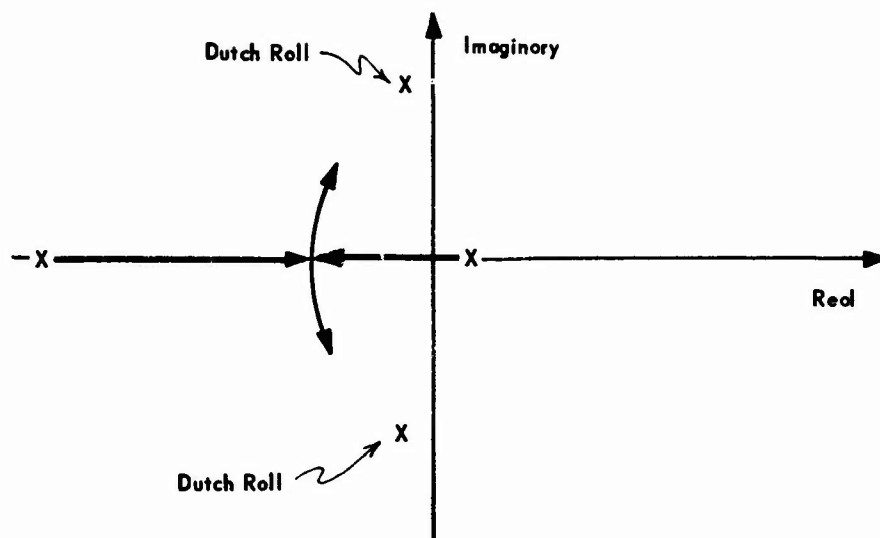


Figure 6.28 Coupled Roll Spiral Mode

At least two inflight experiences with this mode have been documented, and suffice to say that a coupled roll spiral mode causes significant piloting difficulties. One occurrence involved the M2-F2 lifting body, and a second involved the Flight Dynamics Lab variable-stability T-33. Some designs of V/STOL aircraft have indicated that these aircraft would exhibit a coupled roll spiral mode in a portion of their flight envelope<sup>3</sup>. Some pilot comments from simulator evaluations are "rolly," "requires tightly closed roll control loop," or "will roll on its back if you don't watch it."

A coupled roll spiral mode can result from a high value for  $C_{l\beta}$  and a low value for  $C_{lp}$ . The M2-F2 lifting body did in fact possess a high dihedral effect and a quite low roll damping. Examination of the equations for the roll mode and spiral mode characteristic equation roots shows how the root locus shown in figure 6.25 could result as  $C_{lp}$  decreases in absolute magnitude and  $C_{\lambda\beta}$  increases.

## ■ 6.10 STABILITY DERIVATIVES

### ● 6.10.1 INTRODUCTION

Some of the stability derivatives are particularly pertinent in the study of the dynamic modes of aircraft motion, and the more important ones appearing in the functional equations which characterize the dynamic modes of motion should be understood.  $C_{M_q}$ ,  $C_{M_\alpha}$ ,  $C_{l_p}$ ,  $C_{l_\beta}$ ,  $C_{N_r}$ , and  $C_{N_\beta}$  are discussed in the following paragraphs.

<sup>3</sup>AFFDL-TR-65-39, Ground Simulator Elevations of Coupled Roll Spiral Mode Effects on Aircraft Handling Qualities, F. D. Newell, March 1965.

## ● 6.10.2 PARTICULAR STABILITY DERIVATIVES

### ● 6.10.2.1 $C_{M_\alpha}$

This stability derivative is the change in pitching moment coefficient with varying angle of attack and is commonly referred to as the longitudinal static stability derivative. When the angle of attack of the airframe increases from the equilibrium condition, the increased lift on the horizontal tail causes a negative pitching moment about the center of gravity of the airframe. Simultaneously, the increased lift of the wing causes a positive or negative pitching moment, depending on the fore and aft location of the lift vector with respect to the center of gravity. These contributions together with the pitching moment contribution of the fuselage are combined to establish the derivative  $C_{M_\alpha}$ .

The magnitude and sign of the total  $C_{M_\alpha}$  for a particular airframe configuration are thus a function of the center of gravity position, and this fact is very important in longitudinal stability and control. If the center of gravity is ahead of the neutral point, the value of  $C_{M_\alpha}$  is negative, and the airframe is said to possess static longitudinal stability. Conversely, if the center of gravity is aft of the neutral point, the value of  $C_{M_\alpha}$  is positive, and the airframe is then statically unstable.  $C_{M_\alpha}$  is perhaps the most important derivative as far as longitudinal stability and control are concerned. It primarily establishes the natural frequency of the short period mode, and is a major factor in determining the response of the airframe to elevator motions and to gusts. In general, a large negative value of  $C_{M_\alpha}$  (i.e., large static stability) is desirable for good flying qualities. However, if it is too large, the required elevator effectiveness for satisfactory control may become unreasonably high. A compromise is thus necessary in selecting a design range for  $C_{M_\alpha}$ . Design values of static stability are usually expressed not in terms of  $C_{M_\alpha}$  but rather in terms of the derivative  $CMC_L$ , where the relation is:  $C_{M_\alpha} = CMC_L C_{L_\alpha}$ . It should be pointed out that  $CMC_L$  in the above expression is actually a partial derivative for which the forward speed remains constant.

### ● 6.10.2.2 $C_{M_q}$

The stability derivative  $C_{M_q}$  is the change in pitching moment coefficient with varying pitch velocity and is commonly referred to as the pitch damping derivative. As the airframe pitches about its center of gravity path, the angle of attack of the horizontal tail changes, and a lift force is developed on the horizontal tail producing a negative pitching moment on the airframe and hence a contribution to the derivative  $C_{M_q}$ . There is also a contribution to  $C_{M_q}$  because of various "deadweight" aeroelastic effects. Since the airframe is moving in a curved flight path due to its pitching, a centrifugal force is developed on all the components of the airframe. The force can cause the wing to twist as a result of the dead weight moment of overhanging nacelles, and can cause the horizontal tail angle of attack to change as a result of fuselage bending due to the weight of the tail section. In low speed flight,  $C_{M_q}$  comes mostly from

the effect of the curved flight path on the horizontal tail and its sign is negative. In high speed flight the sign of  $CM_q$  can be positive or negative, depending on the nature of the aeroelastic effects. The derivative  $CM_q$  is very important in longitudinal dynamics because it contributes a major portion of the damping of the short period mode for conventional aircraft. As pointed out, this damping effect comes mostly from the horizontal tail. For tailless aircraft, the magnitude of  $CM_q$  is consequently small; this is the main reason for the usually poor damping of this type of configuration.  $CM_q$  is also involved to a certain extent in the damping of the phugoid mode. In almost all cases, high negative values of  $CM_q$  are desired. In the light of the present design trend toward larger radii of gyration in pitch and high altitude flight, it is believed that consideration of  $CM_q$  is necessary in the preliminary design stage.

#### ● 6.10.2.3 $C_{l\beta}$

This stability derivative is the change in rolling moment coefficient with variation in sideslip angle and is usually referred to as the "effective dihedral derivative." When the airframe sideslips, a rolling moment is developed because of the dihedral effect of the wing and because of the usual high position of the vertical tail relative to the equilibrium x-axis. No general statements can be made concerning the relative magnitudes of the contributions to  $C_{l\beta}$  from the vertical tail and from the wing since these contributions vary considerably from airframe to airframe and for different angles of attack of the same airframe.  $C_{l\beta}$  is nearly always negative in sign, signifying a negative rolling moment for a positive sideslip.

The derivative  $C_{l\beta}$  is very important in lateral stability and control, and it is therefore usually considered in the preliminary design of an airframe. It is involved in damping both the Dutch roll mode and the spiral mode. It is also involved in the maneuvering characteristics of an airframe, especially with regard to lateral control with the rudder alone near stall.

#### ● 6.10.2.4 $C_{lp}$

The stability derivative,  $C_{lp}$ , is the change in rolling moment coefficient with change in rolling velocity and is usually known as the roll damping derivative. When the airframe rolls at an angular velocity  $p$ , a rolling moment is produced as a result of this velocity; this moment opposes the rotation.  $C_{lp}$  is composed of contributions, negative in sign, from the wing and the horizontal and vertical tails. However, unless the size of the tail is unusually large in comparison with the size of the wing, the major portion of the total  $C_{lp}$  comes from the wing.

The derivative  $C_{lp}$  is quite important in lateral dynamics because essentially  $C_{lp}$  alone determines the damping in roll characteristics of the aircraft. Normally, it appears that small negative values of  $C_{lp}$  are more desirable than large ones because the airframe will respond better to a given aileron input and will suffer fewer flight perturbations due to gust inputs.

#### ● 6.10.2.5 $C_{N_\beta}$

The stability derivative,  $C_{N_\beta}$ , is the change in yawing moment coefficient with variation in sideslip angle. It is usually referred to as the static directional derivative or the "weathercock" derivative. When the airframe sideslips, the relative wind strikes the airframe obliquely, creating a yawing moment,  $N$ , about the center of gravity. The major portion of  $C_{N_\beta}$  comes from the vertical tail, which stabilizes the body of the airframe just as the tail feathers of an arrow stabilize the arrow shaft. The  $C_{N_\beta}$  contribution due to the vertical tail is positive, signifying static directional stability, whereas the  $C_{N_\beta}$  due to body is negative, signifying static directional instability. There is also a contribution to  $C_{N_\beta}$  from the wing, the value of which is usually positive but very small compared to the body and vertical tail contributions.

The derivative  $C_{N_\beta}$  is very important in determining the dynamic lateral stability and control characteristics. Most of the references concerning full-scale flight tests and free-flight wind tunnel model tests agree that  $C_{N_\beta}$  should be as high as possible for good flying qualities. A high value of  $C_{N_\beta}$  aids the pilot in effecting coordinated turns and prevents excessive sideslip and yawing motions in extreme flight maneuvers and in rough air.  $C_{N_\beta}$  primarily determines the natural frequency of the Dutch roll oscillatory mode of the airframe, and it is also a factor in determining the spiral stability characteristics.

#### ● 6.10.2.6 $C_{N_r}$

The stability derivative  $C_{N_r}$  is the change in yawing moment coefficient with change of yawing velocity. It is known as the yaw damping derivative. When the airframe is yawing at an angular velocity  $r$ , a yawing moment is produced which opposes the rotation.  $C_{N_r}$  is made up of contributions from the wing, the fuselage, and the vertical tail, all of which are negative in sign. The contribution from the vertical tail is by far the largest, usually amounting to about 80 or 90 percent of the total  $C_{N_r}$  of the airframe.

The derivative  $C_{N_r}$  is very important in lateral dynamics because it is the main contributor to the damping of the Dutch roll oscillatory mode. It also is important to the spiral mode. For each mode, large negative values of  $C_{N_r}$  are desired.

### ■ 6.11 PILOT ESTIMATION OF SECOND ORDER MOTION

Pilot-observed data can be used to obtain approximate values for the damped frequency and damping ratio for second order motion such as the short period or Dutch roll.

#### ● 6.11.1 ESTIMATION OF $\omega_d$

To obtain a value for  $\omega_d$ , the pilot needs merely to observe the number of cycles that occur during a particular increment of time.

Then,

$$f_d = \frac{\text{Number of Cycles}}{\text{Time Increment}} = \text{cycles/sec} \quad (6.47)$$

And

$$\omega_d = \left( f_d \frac{\text{cycles}}{\text{sec}} \right) \left( \frac{2\pi \text{ radians}}{\text{cycle}} \right) = \text{radians/sec}$$

The number of cycles can be estimated either by counting peaks or zeroes of the appropriate variable. For short period motion, perturbed  $\theta$  is easily observed, and if counting zeroes is applied to the motion shown in figure 6.29 the result is

$$f_d = \frac{\frac{1}{2} (5 - 1)}{4} = .5 \text{ cycles/sec}$$

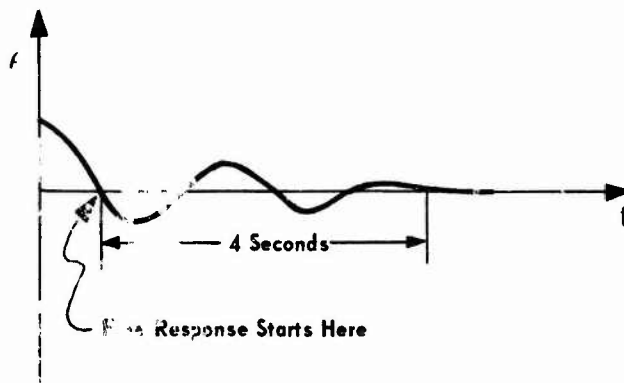


Figure 6.29 Second Order Motion

If zeroes are counted, then

$$f_d = \frac{\frac{1}{2} (\text{number of zeroes} - 1)}{(\text{Time Increment})} \text{ cycles/sec}$$

#### ● 6.11.2 ESTIMATION OF $\zeta$

The pilot can obtain an estimated value for  $\zeta$  by noting the number of peaks that exist during second order motion and using the approximation

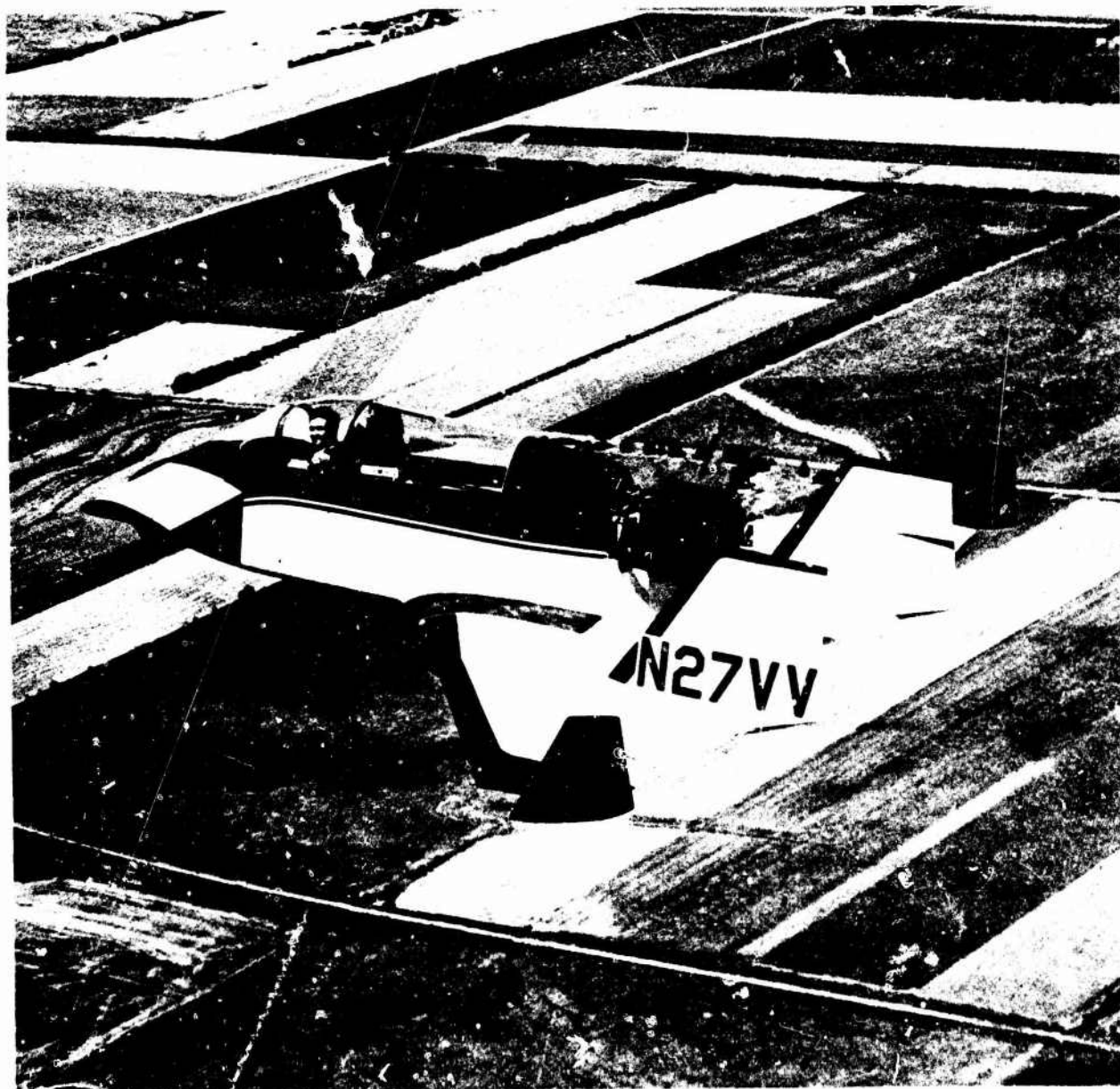
$$\zeta \approx \frac{1}{10} (7 - \text{Number of Peaks}) \quad (6.48)$$

for  $.1 < \zeta < .7$

The motion shown in figure 6.29 thus has an approximate value

$$\zeta \approx \frac{1}{10} (7 - 4) = .3$$

Note that the peaks which occur during aircraft free response are the ones to be used in equation 6.48. If zero observable peaks exist during a second order motion, the best estimate for the value of  $\zeta$  is then "heavily damped, .7 or greater." If seven or more peaks are observed, the best estimate for the value of  $\zeta$  is "lightly damped, .1 or less."



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**VOLUME II**

**CHAPTER VII**

**POST-STALL GYRATIONS/SPINS**

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# **LIST OF ABBREVIATIONS AND SYMBOLS USED IN THIS CHAPTER**

<u>Item</u>	<u>Definition</u>
AOA	angle of attack
b	wing span
$C_D$	drag coefficient
cg	center of gravity
$C_L$	coefficient of lift
$C_l$	rolling moment coefficient
$C_m$	pitching moment coefficient
$C_{m,b}$	pitching moment coefficient non-dimensionalizing with b
$C_n$	yawing moment coefficient
D	drag
g	acceleration equal to that of gravity
H	angular momentum
$\hat{i}$	unit vector
$\hat{j}$	unit vector
$\hat{k}$	unit vector
K	radius of gyration
L	rolling moment
M	pitching moment
m	mass
N	yawing moment
p	roll rate
PSG	post-stall gyration
q	pitch rate
r	yaw rate
r	radius
S	wing area
sec	second
T	torque
V	true velocity
W	weight

## Symbols

$\alpha$	angle of attack
$\alpha_s$	stall angle of attack

<u>Item</u>	<u>Definition</u>
$\beta$	angle of sideslip
$\Delta$	small increment
$\eta$	angle of inclination of the flight-path from the vertical
$\theta$	pitch angle
$\mu$	relative aircraft density
$\rho$	density
$\rho$	air density
$\dot{\phi}$	engine rotation rate
$\phi$	bank angle
$\Omega$	angular velocity
$\omega$	angular velocity

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# CHAPTER POST-STALL GYRATIONS/SPINS

# VII

## ■ 7.1 INTRODUCTION

"Of the myriad of coupled motions which an airplane can perform, the spin stands out as being unique. When an airplane is stalled and left to itself, it will perform some sort of rolling, yawing and pitching motion which, if allowed to continue, may develop into a characteristic motion called a spin, in which the airplane descends rapidly toward the earth in a helical movement about the vertical axis at an angle of attack between the stall and  $90^\circ$ ." (Reference 1, page 2)

From this classical description it is clear that an aircraft spin is an extremely complex maneuver simultaneously involving pitch, roll, and yaw rates along with extreme angles of attack and large angles of sideslip. It is indeed more complex than the description. In the initial stages, the aircraft will still have some of its translational velocity, and the aircraft can spin at angles of attack greater than  $90^\circ$ .

Recently a renewed interest in the high angle of attack (AOA) flight regime has generated considerable interest in designing to avoid the spin entirely in tactical aircraft. Clearly such design goals are worthy, and a whole set of terms (not necessarily new) has been redefined to make more explicit the requirements which hopefully will make spin resistant tactical aircraft a reality. Words like "departure," stall, "post-stall gyration (PSG)," and "spin" itself have taken on different shades of meaning since the publication of reference 2. Given the complicated motions associated with a PSG or a spin and the explicit requirements now imposed by references 2 and 3, it is imperative that the test pilot clearly understand the precise terminology of the high AOA flight regime.

### ● 7.1.1 DEFINITIONS

#### ● 7.1.1.1 Stall Versus Out-of-Control.

Stalls and associated aerodynamic phenomena have been described completely in chapter 2, but it is worth repeating the formal definition of a stall from page 67 of reference 3. In terms of angle of attack, the stall is defined as the lowest of the following:

- a. Angle of attack for the highest steady load factor, normal to the flightpath, that can be attained at a given speed or Mach number.

- b. Angle of attack, for a given speed or Mach number, at which abrupt or uncontrollable pitching, rolling, or yawing occurs. Angular limits of 20 degrees (Classes I, II, or III) or 30 degrees (Class IV) are specified in paragraph 3.4.2.1.2 of reference 3.
- c. Angle of attack, for a given speed or Mach number, at which intolerable buffeting is encountered.
- d. An arbitrary angle of attack, allowed by paragraph 3.1.9.2.1 of reference 3, which may be based on such considerations as ability to perform altitude corrections, excessive sinking speed, or ability to execute a go-around.

Reference 2 defines the stall angle of attack more simply: The angle of attack for maximum usable lift at a given flight condition. This latter definition is the one most useful in this course, but the student must understand that "maximum usable lift" is determined from one of the four conditions given above.

#### 7.1.1.2 Departure.

Departure is defined as that event in the post-stall flight regime which precipitates entry into a PSG, spin or deep stall condition (reference 2, paragraph 6.3.9). Notice two things about this definition. First, departure occurs in the post-stall flight regime; that is, the stall always precedes departure. It can be inferred then that the angle of attack for maximum usable lift is always less than the angle of attack at which departure occurs. The second point is that only one of three motions may result after departure - the aircraft enters either a PSG, spin or deep stall (of course, a PSG can progress into a spin or deep stall). Implicit in this definition is the implication that an immediate recovery cannot be attained. For example, a light aircraft whose stall is defined by a g-break, may recover immediately if the longitudinal control pressure is relaxed. However, note that movement or position of controls is not mentioned in the definition. The same light aircraft that would not depart if control pressures were relaxed at the stall may depart and enter a spin if pro-spin controls are applied at the stall. Hence, in discussing susceptibility or resistance to departure one must specify control positions as well as loading and configuration.

The departure event is usually a large amplitude, uncommanded, and divergent motion. Such descriptive terms as nose slice or pitch-up are commonly used to describe the event. Large amplitude excursions imply changes in yaw, roll, or pitch greater than 20 degrees (class I, II, and III) or 30 degrees (class IV) (reference 3, paragraph 3.4.2.1.2). Uncommanded motions are motions not intended by the pilot, even though the control positions are legitimately causing the departure. The aircraft may not follow the pilot's commands for a number of reasons: the high angle of attack may render the control surface ineffective when moved to its desired position; or the pilot may be unable to position the stick to put the surface in the desired position due to lateral or transverse g loads. In either of these conditions the aircraft motion is "uncommanded." Finally, a divergent motion is one which either continuously or periodically increases in amplitude. The T-33 usually exhibits a "bucking" motion after the stall in which the nose periodically rises and falls. However, the motion is not divergent unless aggravated by full aft stick or some other pro-spin control. The T-38 will sometimes

exhibit a non-divergent lateral oscillation near the stall angle of attack. Neither of these motions are normally counted as departures, though their occurrence does serve as warning of impending departure if further misapplications of controls are made. With this sort of background it is easy to see why a departure is so hard to define, yet is relatively easy for a pilot to recognize. Next one must examine the terms "post-stall gyration", "spin" and "deep stalls", used to define a departure.

#### ● 7.1.1.3 Post-Stall Gyration.

A post-stall gyration is an uncontrolled motion about one or more axes following departure (reference 2, paragraph 6.3.10). PSG is a very difficult term to define concisely because it can occur in so many different ways. Frequently, the motions are completely random about all axes and no more descriptive term than PSG can be applied. On the other hand a snap roll or a tumble are post-stall gyrations. The main difficulty lies in distinguishing between a PSG and either the incipient phase of a spin or an oscillatory spin. The chief distinguishing characteristic is that a PSG may involve angles of attack that are intermittently below the airplane's stall angle of attack, whereas a spin always occurs at angles of attack greater than stall.

#### ● 7.1.1.4 Spin.

A spin is a sustained yaw rotation at angles of attack above the aircraft's stall angle of attack (reference 2, paragraph 6.3.11). This definition bears a bit of explanation in that a spin is certainly not altogether a yaw rotation. Only the perfect flat spin ( $\alpha = 90$  degrees) could satisfy that constraint. The inference is, however, that the yaw rotation is dominant in characterizing a spin. Indeed, to a pilot, the recognition of a sustained (though not necessarily steady) yaw rate is probably the most important visual cue that a spin is occurring. Even though roll rate and yaw rate are often of nearly the same magnitude, the pilot still ordinarily recognizes the spin because of the yaw rate. In steep spins (with  $\alpha$  relatively close to  $\alpha_s$ ) it is quite easy to confuse the roll rate and yaw rate and pilots sometimes have difficulty in recognizing this type of motion and treating it as a spin. The steep inverted spin is particularly confusing since the roll and yaw rates are in opposite directions. Once again though, the yaw rate determines the direction of the spin and the required control manipulations to recover. All in all, it is well to remember that the spin is truly a complicated maneuver involving simultaneous roll, pitch, and yaw rates and high angles of attack. And, even though the overall rotary motion in a spin will probably have oscillations in pitch, roll, and yaw superimposed upon it, it is still most easily recognized by its sustained yawing component.

#### ● 7.1.1.5 Deep Stall.

A deep stall is an out-of-control flight condition in which the airplane is sustained at an angle of attack well beyond that for  $\alpha_s$  while experiencing negligible rotational velocities (reference 2, paragraph 6.3.12). It may be distinguished from a PSG by the lack of significant motions other than a high rate of descent. The deep stall may be a fairly stable maneuver such as a falling leaf, or it can be characterized by large amplitude angle of attack oscillations. For an aircraft to stay in a deep stalled condition, significant oscillations

must be limited to the longitudinal axis. Lateral and directional control surfaces are either stalled or blanked out. Depending on the pitching moment coefficient, recovery may or may not be possible.

#### ●7.1.2 SUSCEPTIBILITY AND RESISTANCE TO DEPARTURES AND SPINS:

Susceptibility/resistance to departures and spins has become an extremely important design goal for the generation of high performance aircraft presently in the design stage. Reference 4 offers convincing proof that such design emphasis is in fact overdue. But, for the designer to meet this requirement in an aircraft and for the test pilot to test against this requirement, it is essential that the words "susceptible" and "resistant" be understood alike by all concerned.

##### ●7.1.2.1 Extremely Susceptible to Departure (Spins).

An aircraft is said to be extremely susceptible to departure (spins) if the uncontrolled motion occurs with the normal application of pitch control alone or with small roll and yaw control inputs. The only allowable roll and yaw control inputs are those normally associated with a given maneuver task. In short, an airplane that departs or enters a spin during Phase A of the flight test demonstration falls within this category (reference 2, paragraph 3.4.1.8).

##### ●7.1.2.2 Susceptible to Departure (Spins).

An aircraft is said to be susceptible to departure (spins) when the application or brief misapplication of pitch and roll and yaw controls that may be anticipated in normal operational use cause departure (spin). The amount of misapplied controls to be used will be approved by the procuring activity for Phase B of the flight test demonstration. In other words each aircraft will be stalled and aggravated control inputs will be briefly applied to determine departure (spin) susceptibility.

##### ●7.1.2.3 Resistant to Departure (Spins).

An aircraft is said to be departure (spin) resistant if only large and reasonably sustained misapplication of controls results in a departure (spin). "Reasonably sustained" means up to 3 seconds before recovery is initiated (reference 2, table I). This time delay may be increased for aircraft without positive indication of impending loss of control. This aircraft departs (spins) during Phase C of the flight test demonstration.

##### ●7.1.2.4 Extremely Resistant to Departure (Spins).

An aircraft is said to be extremely resistant to departure (spins) if these motions occur only after abrupt, inordinately sustained application of gross, abnormal, pro-departure controls. Aircraft in this category will only depart (spin) in Phase D of the flight test demonstration when the controls are applied and held in the most critical manner to attain each possible mode of post-stall motion and held for various lengths of time up to 15 seconds or three spin turns, whichever is longer.

### ●7.1.3 SPIN MODES:

Adjective descriptors are used to describe general characteristics of a given spin and these adjectives specify the spin mode. Average values of angle of attack, for example, would allow categorization of the spin as either upright (positive angle of attack) or inverted (negative angle of attack). An average value of angle of attack would also allow classification of a spin as either flat (high angle of attack) or steep (lower angle of attack). Finally, the average value of the rotational rate compared with the oscillations in angular rates about all three axes determines the oscillatory character of the spin. One descriptive modifier from each of these groups may be used to specify the spin mode.

Table I

SPIN MODE MODIFIERS

Group 1	Group 2	Group 3
Upright	Steep	Smooth Mildly Oscillatory
Inverted	Flat	Oscillatory Highly Oscillatory Violently Oscillatory

The most confusing thing about mode identification is the proper use of group 2 and group 3 modifiers. Perhaps the following tabulated data, extracted from reference 5, will provide insight for understanding how to use these terms.

Table II

F-4E SPIN MODES

Mode	Average AOA (deg)	AOA Oscillations (deg)	Yaw Rate (deg/sec)	Roll Rate (deg/sec)	Pitch Rate (deg/sec)
Steep-Smooth	42	+5	40-50	50	15
Steep-Mildly Oscillatory	45-60	+10	45-60	--	--
Steep-Oscillatory	50-60	+20	50-60 (with large oscillations)	Same as yaw rate	--
Flat-Smooth	77-80	Negligible	80-90	25	7

Note: One mode reported in reference 5 has been omitted from this table because the terminology did not fully conform to that of reference 2. It was called "highly oscillatory" with angle of attack excursions of +30 degrees.

#### ●7.1.4 SPIN PHASES:

A typical spin may be divided into the phases shown in figure 7.1

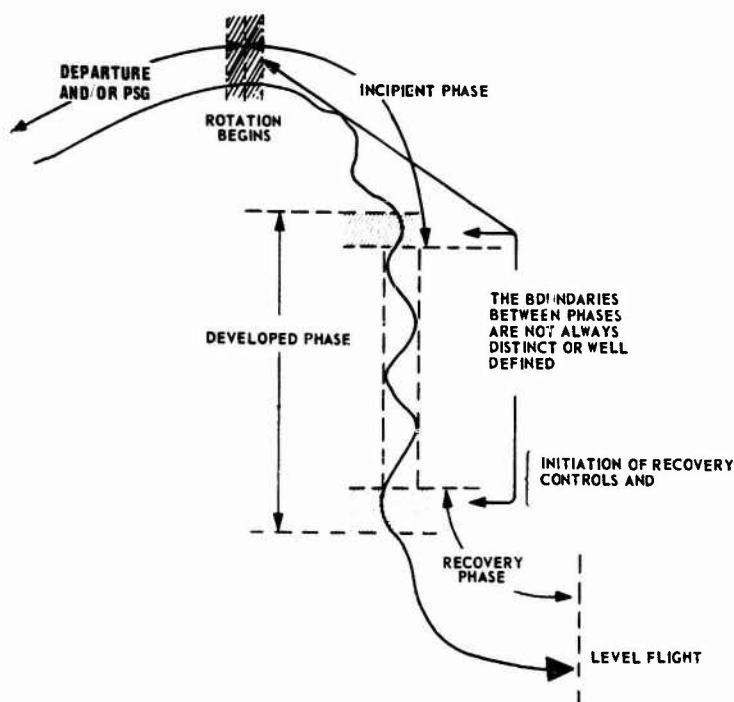


FIGURE 7.1 SPIN PHASES

##### ●7.1.4.1 Incipient Phase.

The incipient phase of a spin is the initial, transitory part of the motion during which it is impossible to identify the spin mode. However, notice in figure 7.1 that the yaw rotation begins as the incipient phase begins; that is, the visual cue to the pilot is of a sustained (though by no means steady) yaw rotation. A further distinction between the PSG (if one occurs) and the incipient phase of the spin is that the angle of attack is continuously above the stalled angle of attack ( $\alpha_s$ ) for the aircraft, even in the incipient phase of the spin. During a PSG the angle of attack may intermittently be less than  $\alpha_s$ . This incipient phase continues until a recognizable spin mode develops, another boundary very difficult to establish precisely. In fact the test pilot may not recognize such a mode until he has seen it several times; but careful examination of data traces and film may reveal that a "recognizable" mode had occurred. In this case "recognizable" does not necessarily mean recognizable in flight, but distinguishable to the engineer from all available data. In short, the incipient phase of the spin is a transitory motion easily confused with a PSG, but distinctly different from either a PSG or the developed phase of the spin.

#### ● 7.1.4.2 Developed Phase.

The developed phase of a spin is that stage of the motion in which it is possible to identify the spin mode. During this phase it is common for oscillations to be present, but the mean motion is still abundantly clear. The aerodynamic forces and moments are not usually completely balanced by the corresponding linear and angular accelerations, but at least equilibrium conditions are being approached. Generally it is evident in the cockpit that the developed phase is in progress, though the exact point at which it began may be quite fuzzy. Since the aircraft motion is approaching an equilibrium state, it is frequently advisable to initiate recovery before equilibrium is achieved. For example, during the T-38 test program warning lights were installed to signal a buildup in yaw rate. Test pilots initiated recovery attempts when these lights came on. Still, in the flat spin mode with recovery initiated at 85 degrees per second, a peak yaw rate of 165 degrees per second was achieved. The longitudinal acceleration at the pilot's station was approximately 3.5 g and the spin was terminated by deployment of the spin chute (reference 6, pages 10, 11). The developed spin, while it may be more comfortable due to less violent oscillations, can be deceptively dangerous, and the spin phase which follows can be disastrous.

#### ● 7.1.4.3 Fully Developed Phase.

A fully developed spin is one in which the trajectory has become vertical and no significant change in the spin characteristics is noted from turn to turn. Many aircraft never reach this phase during a spin, but when they do, they are often very difficult to recover. The smooth, flat spin of the F-4 is a classic example in which this phase is attained and from which there is no known aerodynamic means of recovery. But a fully developed spin obviously requires time and altitude to be generated; how much time and how much altitude are strong functions of entry conditions. As a general rule, departures that occur at high airspeeds (high kinetic energy) require more time and altitude to reach the fully developed phase than departures which occur at low kinetic energy. Finally, the spin characteristics which remain essentially unchanged in the fully developed phase include such parameters as time per turn, body axis angular velocities, altitude loss per turn, and similar quantities. However, the definition does not prohibit a cyclic variation in any of these parameters. Hence, a fully developed spin can be oscillatory.

With this rather lengthy set of definitions in mind it is now appropriate to look more closely at spinning motions and at the aerodynamic and inertial factors which cause them and the PSG.

### ■ 7.2 THE SPINNING MOTION

Because the PSG is a random and usually a highly irregular motion, it is very difficult to study. On the other hand, the spin can approach an equilibrium condition and is therefore much more easily understood. Further, since the PSG is affected by the same aerodynamic and mass loading characteristics as the spin, an understanding of the spin and the factors affecting it are appropriate to the purposes of this course.

### ●7.2.1 DESCRIPTION OF FLIGHTPATH

An aircraft spin is a coupled motion at extreme attitudes that requires all six equations of motion for a complete analysis. It is usually depicted with the aircraft center of gravity describing a helical path as the airplane rotates about an axis of rotation. Figures 7.2 shows such a motion. Notice that the spin axis of rotation may be curved and that the spin vector  $\omega$  is constantly changing. Such a motion is highly complex, but by making some approximations a simplification results which can be very useful in understanding the spin and its causes.

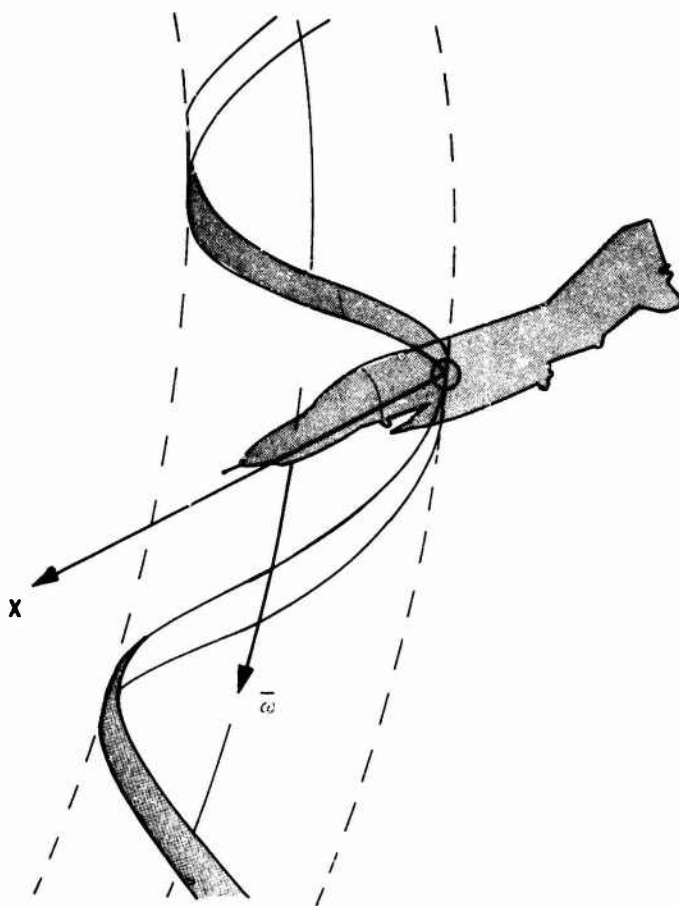
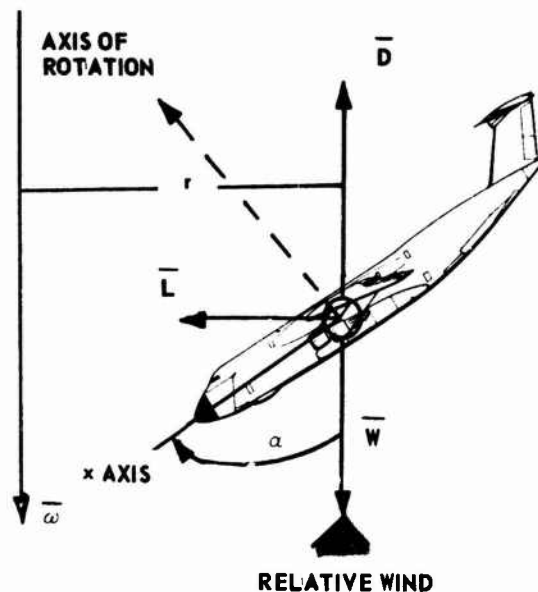


FIGURE 7.2 HELICAL SPIN MOTION

In a fully developed spin with no sideslip the spin axis is vertical as indicated in figure 7.3.



**FIGURE 7.3 FORCES IN A STEADY SPIN WITHOUT SIDESLIP**

If one ignores the side force, the resultant aerodynamic force acts in the x-z plane and is approximately normal to the wing chord. Taking the relative wind to be nearly vertical, a summation of vertical forces gives:

$$W = D = \frac{1}{2} \rho V^2 S C_D \quad (7.1)$$

A similar summation of horizontal forces suggests that the lift component balances the so-called centrifugal force so that

$$mr\omega^2 = L = \frac{1}{2} \rho V^2 S C_L \quad (7.2)$$

Equation 7.1 suggests that as AOA increased (and  $C_D$  increased) the rate of descent ( $V$ ) must decrease. Furthermore, at a stalled AOA,  $C_L$  decreases as AOA increases. With these two facts in mind it is clear that the left hand side of equation 7.2 must decrease as the AOA increases in a spin. The rotation rate,  $\omega$ , as will be shown later, tends to increase as AOA increases; hence, the radius of turning  $r$  must decrease rapidly

as AOA increases. These observations point up the fact that in a fully developed spin  $\omega$  and the relative wind are parallel and become more nearly coincident as the AOA increases. In fact the inclination ( $\eta$ ) of the flightpath (relative wind) to the vertical is given by

$$\tan \eta = \frac{r\omega}{V}$$

A typical variation of  $\eta$  with AOA is from about 5.5 degrees at  $\alpha = 50$  degrees to 1 degree at  $\alpha = 80$  degrees (reference 7, page 533). So, it is not farfetched to assume that  $\omega$  is approximately parallel to the relative wind in a fully developed spin.

All of these observations have been made under the assumption that the wings are horizontal and that sideslip is zero. These effects, while extremely important, are beyond the scope of this course, but references 7 and 8 offer some insight into them. It is also noteworthy that this simplified analysis is valid only for a fully developed spin. However, the trends to be noted and an understanding of the underlying physical phenomena will give the student a greater appreciation of the other phases of the spin and of the post-stall gyration.

### ● 7.2.2 AERODYNAMIC FACTORS

In the post-stall flight regime the aircraft is affected by very different aerodynamic forces than those acting upon it during unstalled flight. Many aerodynamic derivatives change sign; others which are insignificant at low angles of attack become extremely important. Probably the most important of these changes is a phenomenon called autorotation which stems largely from the post-stall behavior of the wing.

#### ● 7.2.2.1 Autorotative Couple of the Wing.

If a wing is operating at  $\alpha_1$  (low angle of attack) and experiences a  $\Delta\alpha$  due to wing drop, there is a restoring moment from the increased lift. If, on the other hand, a wing operating at  $\alpha_2$  ( $\alpha_2 > \alpha_s$ ) experiences a sudden drop, there is a loss of lift and an increase in drag that tends to prolong the disturbance and sets up autorotation. These aerodynamic changes are illustrated in figure 7.4.

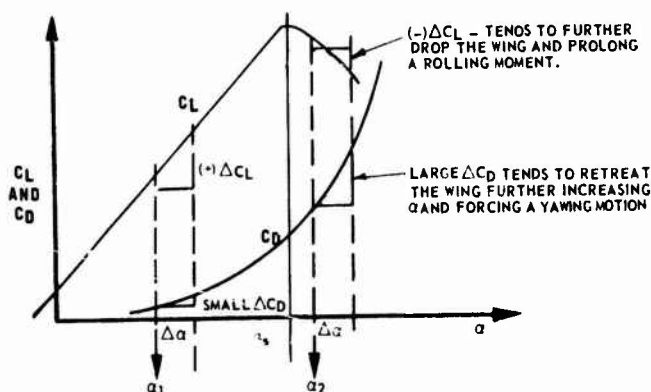


FIGURE 7.4 CHANGES IN  $C_L$  AND  $C_D$  WITH  $\alpha < \alpha_s$  AND  $\alpha > \alpha_s$

Consider now a wing flying in the post-stall region of figure 7.4 and assume that some disturbance has given that wing an increase in  $\alpha$  which tends to set up a yawing and rolling motion to the right as shown in figure 7.5. The angle of attack of the advancing wing (section A) corresponds to  $\alpha_2$  in figure 7.4 while the angle of attack of the retreating wing (section R) corresponds to  $\alpha_2 + \Delta\alpha$  in figure 7.4. Figure 7.6 shows these two sections and illustrates why the advancing wing is operating at a lesser angle of attack than the retreating wing. In each case the velocity vectors are drawn as they would be seen by an observer fixed to the respective wing section. The difference in resultant aerodynamic force  $R_A - R_R$  acting at section A will, in general, be a force  $\Delta\bar{F}$ , depicted in figure 7.7. Notice that  $\Delta F_x$  is in a positive x-direction, while  $\Delta F_z$  is in a negative z-direction.  $\Delta F_x$  forms a couple as depicted in figure 7.8 which tends to sustain the initial yawing moment to the right. Of course,  $\Delta F_z$  contributes a similar rolling couple about the x-axis which tends to sustain the initial rolling moment to the right. Ordinarily, the autorotative couples generated by the wing are the most important aerodynamic factors causing and sustaining a spin. However, the other parts of the aircraft also have a part to play.

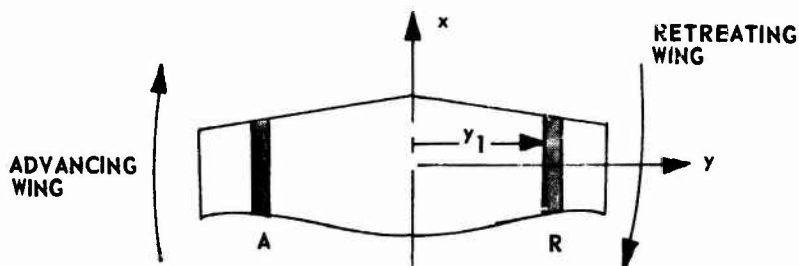


FIGURE 7.5 PLAN VIEW OF AUTOROTATING WING

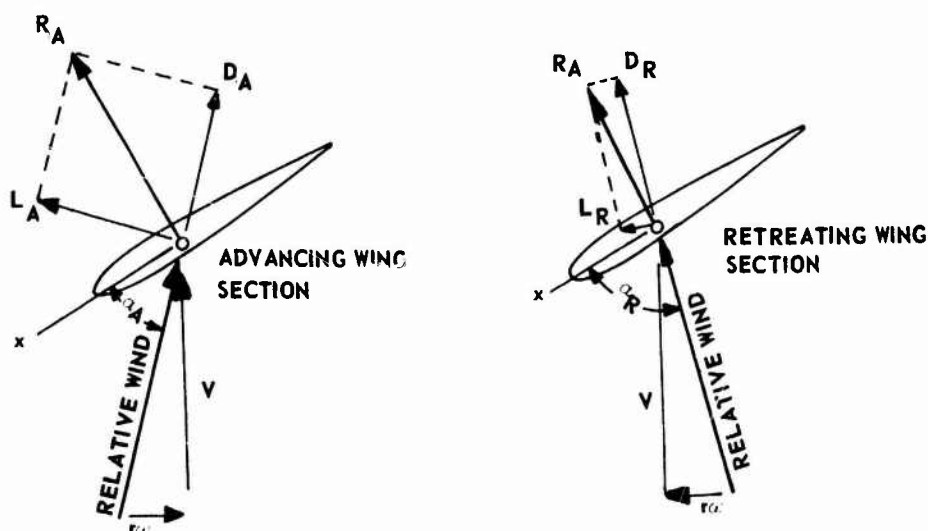


FIGURE 7.6 DIFFERENCE IN AOA FOR THE ADVANCING AND RETREATING WING IN AUTOROTATION

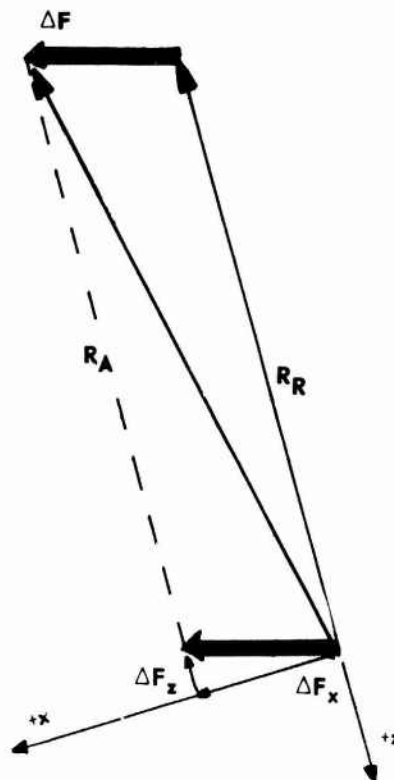


FIGURE 7.7 COMPONENTS OF DIFFERENCE IN RESULTANT AERODYNAMIC FORCES (ADVANCING WING)

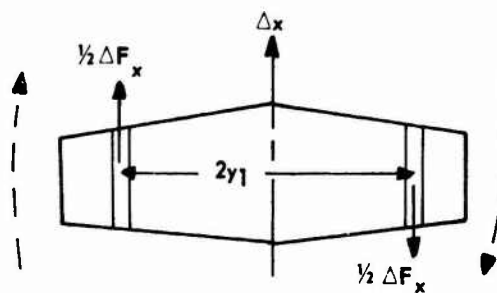


FIGURE 7.8 AUTOROTATIVE YAWING COUPLE

#### ● 7.2.2.2 Fuselage Contribution.

The aerodynamic forces on the fuselage at stalled angle of attack are very complex, are highly dependent on fuselage shape, and may either oppose or increase the autorotative couples. Sidewash flow over the fuselage greatly affects the dihedral effect ( $C_{l_\beta}$ ) and may even increase it to values greater than those observed for unstalled flight (reference 7, page 529). Weathercock stability ( $C_{n_\beta}$ ) will also be affected significantly by sidewash flow over the fuselage. As an example of the possible

contributions of the fuselage to the autorotative couple consider the effects of fuselage shape as illustrated in figure 7.9. The fuselage in figure 7.9a acts much like an airfoil section and may well generate a resultant aerodynamic force which would contribute to the yawing autorotative couple. Of course the fuselage shape will determine the relative sizes of "lift" and "drag" contributed by the rotating nose section. A box-like fuselage cross-section will probably give a resultant aerodynamic force opposing the yaw autorotation. An extreme example of this type of fuselage cross-section reshaping is the strakes added to the nose of the T-37, as in figure 7.9b. Clearly the flow separation produced by the strakes in a flow field with considerable sidewash reorients the resultant aerodynamic force in such a way as to produce an anti-spin yawing moment. Such devices have also been proposed (and tested) for the F-100, F-106, and F-111.

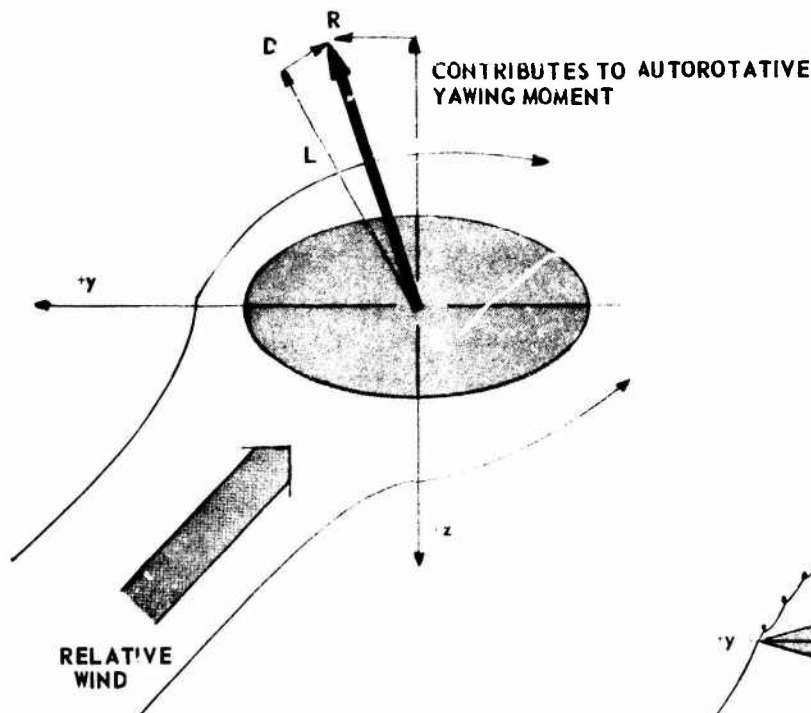


FIGURE 7.9A PLAIN FUSELAGE

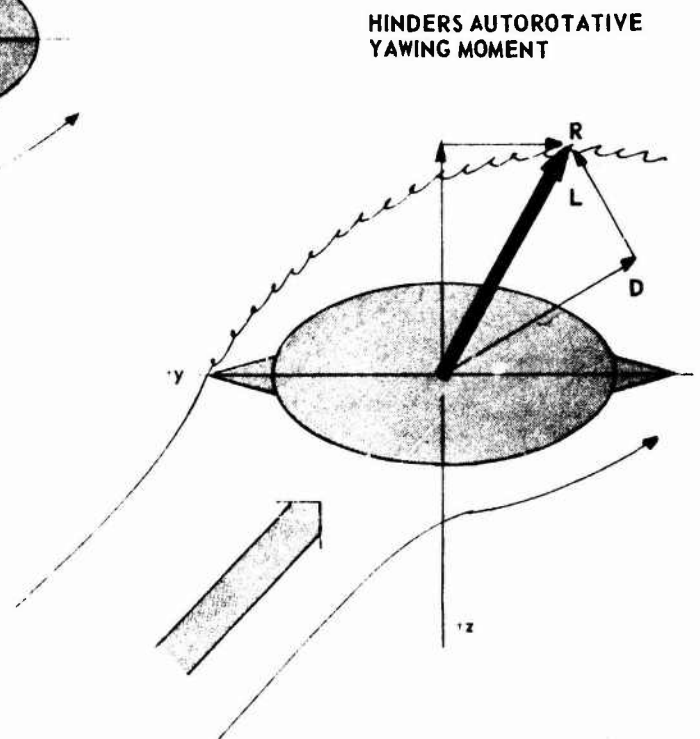


FIGURE 7.9B FUSELAGE WITH STRAKES

### ● 7.2.2.3 Changes in Other Stability Derivatives.

All of the other stability derivatives, especially those depending on the lift curve slope of the wing, behave in a different manner in the post-stall flight regime. However, a fuller discussion of the post-stall behavior of such derivatives as  $C_{l_p}$ ,  $C_{n_p}$ ,  $C_{n_r}$ , and combinations of these derivatives is given in reference 7, page 529. For the purposes of this course it suffices to say that  $C_{l_p}$  becomes positive and  $C_{n_p}$  may become positive in the post-stall flight regime;  $C_{n_r}$  may also become greater in stalled flight. Each of these changes contributes to autorotation, the aerodynamic phenomenon which initiates and sustains a spin. However, aerodynamic considerations are by no means the only factors affecting the post-stall motions of an aircraft. The inertia characteristics are equally important.

### ● 7.2.3 AIRCRAFT MASS DISTRIBUTION:

#### ● 7.2.3.1 Principal Axes.

For every rigid body there exists a set of principal axes for which the products of inertia are zero and one of the moments of inertia is the maximum possible for the body. For a symmetrical aircraft, this principal axis system is frequently quite close to the body axis system. For the purpose of this course, the small difference in displacement is neglected, and the principal axes are assumed to lie along the body axes. Figure 7.10 illustrates what the actual difference might be.

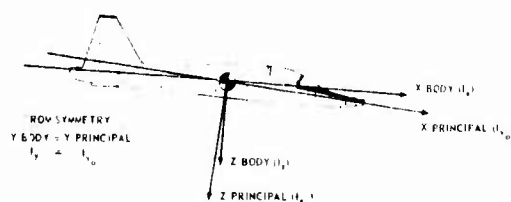


FIGURE 7.10 BODY AND PRINCIPAL AXES PROXIMITY

#### ● 7.2.3.2 Radius of Gyration.

The center of gyration of a body with respect to an axis is a point at such a distance from the axis that, if the entire mass of the body were concentrated there, its moment of inertia would be the same as that of the body. The radius of gyration ( $K$ ) of a body with respect to an axis is the distance from the center of gyration to the axis. In equation form

$$\int (y^2 + z^2) dm = I_x = K_x^2 m$$

$$\int (x^2 + z^2) dm = I_y = K_y^2 m$$

$$\int (x^2 + y^2) dm = I_z = K_z^2 m$$

or

$$K_i = I_i/m, i = x, y, \text{ or } z \quad (7.3)$$

#### ●7.2.3.3 Relative Aircraft Density.

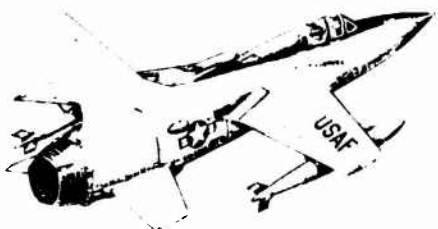
Y:

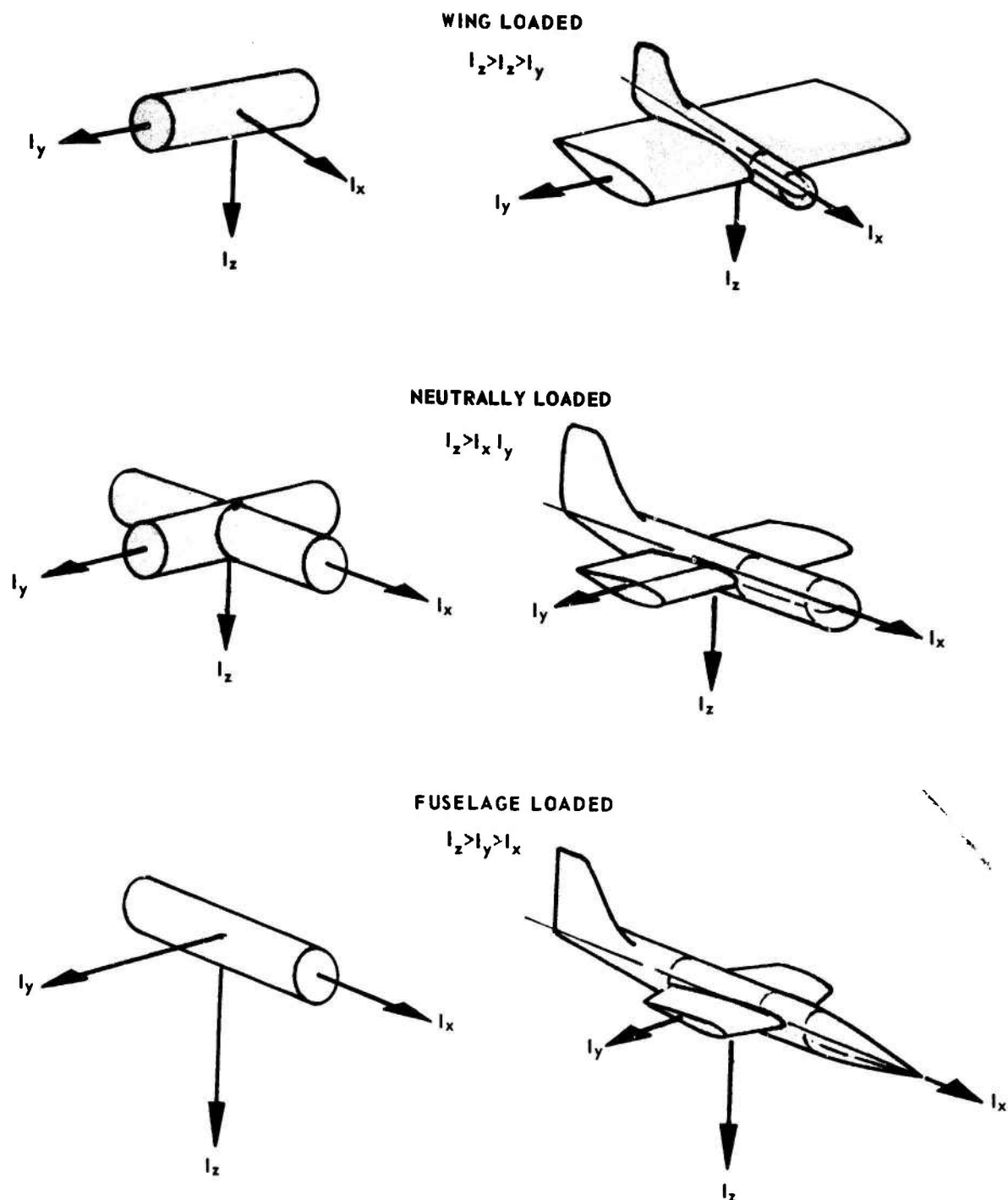
A nondimensional parameter called relative aircraft density ( $\mu$ ) is frequently used to compare aircraft density to air density.

$$\mu = \frac{m/Sb}{\rho} = \frac{m}{\rho Sb} \quad (7.4)$$

#### ●7.2.3.4 Relative Magnitude of the Moments of Inertia.

The aircraft mass distribution is frequently used to classify the aircraft according to loading. Because aircraft are "flattened" into the XY plane,  $I_z$  is invariably the maximum moment of inertia.  $I_x$  is greater or less than  $I_y$  depending on the aircraft's mass distribution. The relative magnitudes of the moments of inertia are shown in figure 7.11. As will be seen in the next paragraph the relative magnitudes of  $I_x$ ,  $I_y$ , and  $I_z$  are of utmost importance in interpreting the equations of motion.





**FIGURE 7.11 AIRCRAFT MASS DISTRIBUTION**

### 7.3 EQUATIONS OF MOTION

Maneuvers within the post-stall flight regime can be analyzed by using all six equations of motion and integrating them numerically on a computer. From such studies, predictions of rate of rotation, angle of attack, magnitude of the oscillations, optimum recovery techniques, and other parameters can be made. However, such studies must use rather inaccurate theory to predict stability derivatives or else depend on wind tunnel data or free flight model tests to provide the aerodynamic data. Hence, many researchers prefer to rely almost completely on model tests for predictions prior to flight tests. Correlation between model tests and aircraft flight tests is generally good. But model tests also have limitations. Spin tunnel tests primarily examine developed or fully developed spins; there is no good way to investigate PSG's or the incipient phase of the spin in the spin tunnel. Reynolds number effects on both spin tunnel and free flight models make it very difficult to accurately extrapolate to the full scale aircraft. Engine gyroscopic effects are not often simulated in model tests. Finally, model tests are always done for a specific aircraft configuration, which is a distinct advantage for a flight test program even though it does not suit the purposes of this course. However, it would be foolish to ignore either computer analyses or model tests in preparing for a series of post-stall flight tests. For obvious reasons, this course will be restricted to a much simplified look at the equations of motion as applied to a fully developed spin.

#### 7.3.1 ASSUMPTIONS:

The analytical treatment used in these notes is based on many simplifying assumptions, but even with these assumptions good qualitative information can be obtained. The most important assumption is that only a fully developed spin with the wings horizontal will be considered. The wings horizontal, fully developed spin involves a balance between applied and inertial forces and moments. Some of the ramifications of this assumption are:

- a. The rate of descent ( $V$ ) is virtually constant, as is altitude loss per turn.
- b.  $\bar{V}$  and  $\bar{\omega}$  are parallel.
- c. The time per turn is constant, or  $\bar{\omega}$  is constant. Hence,  $\dot{p} = \dot{q} = \dot{r} = 0$ .
- d. With the wings horizontal  $\bar{\omega}$  lies entirely within the  $xz$  plane and  $q = 0$ . ( $\bar{\omega} = p \hat{i} + r \hat{k}$ ).

Initially, it will also be assumed that the applied moments consist entirely of aerodynamic ones, although other factors will be considered in later paragraphs.

#### 7.3.2 GOVERNING EQUATIONS:

The reference frame for expressing moments, forces, accelerations, etc., is the  $xyz$  body axis frame which rotates at the same rate as the spin rotation rate  $\omega$ . The origin of the  $xyz$  axes is centered at the

aircraft's cg and translates downward at a rate equal to the constant rate of descent  $\bar{V}$ . With this background the forces acting on the aircraft can be examined.

#### ●7.3.2.1 Forces.

The external forces applied to the aircraft and expressed in an inertial reference frame follow Newton's second law.

$$\bar{F} = m \dot{\bar{V}}$$

Expressing  $\dot{\bar{V}}$  in the xyz reference frame,

$$\bar{F} = m (\dot{\bar{V}} + \bar{\omega} \times \bar{V})$$

But since  $\bar{V}$  is constant in the fully developed spin and since  $\bar{\omega}$  and  $\bar{V}$  are parallel,

$$\bar{F} = 0$$

The elimination of the force equations in this fashion merely reinforces the idea that the rotary motion is the important motion in a spin and one would expect the significant equations to be the moment equations.

#### ●7.3.2.2 Moments.

The moment equations to be considered have already been developed in Chapter I and are repeated below.

$$G_x = \dot{p} I_x + qr (I_z - I_y) - (\dot{r} + pq) I_{xz} \quad (7.5)$$

$$G_y = \dot{q} I_y - pr (I_z - I_x) + (p^2 - r^2) I_{xz} \quad (7.6)$$

$$G_z = \dot{r} I_z + pq (I_y - I_x) + (qr - \dot{p}) I_{xz} \quad (7.7)$$

Utilizing the assumption that the body axes xyz are also principal axes and considering  $G$  to consist of aerodynamic moments only, these equations become:

$$L_{aero} = \dot{p} I_x + qr (I_z - I_y) \quad (7.8)$$

$$M_{aero} = \dot{q} I_y - pr (I_z - I_x) \quad (7.9)$$

$$N_{aero} = \dot{r} I_z + pq (I_y - I_x) \quad (7.10)$$

Solving for the angular accelerations shows the contributions of each type of moment to that acceleration.

$$\dot{p} = \frac{L_{aero}}{I_x} + \frac{I_y - I_z}{I_x} q r \quad (7.11)$$

$$\dot{q} = \frac{M_{aero}}{I_y} + \frac{I_z - I_x}{I_y} p r \quad (7.12)$$

$$\dot{r} = \frac{N_{aero}}{I_z} + \frac{I_x - I_y}{I_z} p q \quad (7.13)$$

aerodynamic term                      inertial term

The body axis angular accelerations can also be expressed in terms of aerodynamic coefficients and the relative aircraft density.

$$\begin{aligned} \frac{L_{aero}}{I_x} &= \frac{\frac{1}{2} \rho V^2 S b}{K_x^2 m} C_\ell = \frac{V^2}{\frac{2m}{\rho S b} K_x^2} C_\ell \\ &= \frac{V^2}{2\mu K_x^2} C_\ell \end{aligned}$$

In a similar manner,

$$\frac{N_{aero}}{I_z} = \frac{V^2}{2\mu K_z^2} C_n$$

It is common practice in post-stall/spin literature to define  $C_m$  on the basis of wingspan instead of on the basis of wing chord as is done in most other stability and control work. This change is made to allow a consistent definition of  $\mu$  and is indicated by a second subscript; that is,  $C_m$  becomes  $C_{m,b}$ . Then,

$$\frac{M_{aero}}{I_y} = \frac{v^2}{2\mu K_y^2} C_{m,b}$$

Equations 7.11 through 7.13 then become

$$\dot{p} = \frac{v^2 C_{\ell}}{2\mu K_x^2} + \frac{I_y - I_z}{I_x} qr \quad (7.14)$$

$$\dot{q} = \frac{v^2 C_{m,b}}{2\mu K_y^2} + \frac{I_z - I_x}{I_y} pr \quad (7.15)$$

$$\dot{r} = \frac{v^2 C_n}{2\mu K_z^2} + \frac{I_x - I_y}{I_z} pq \quad (7.16)$$

With this brief mathematical background it is now appropriate to consider the aerodynamic prerequisites for a fully developed spin to occur.

### ● 7.3.3 AERODYNAMIC PREREQUISITES:

For a fully developed upright spin with the wings horizontal

$\dot{p} = \dot{q} = \dot{r} = q = 0$  and equations 7.14, 7.15, and 7.16 yield

$$C_{\ell} = 0 \quad (7.17)$$

$$-\frac{v^2 C_{m,b}}{2\mu K_y^2} = \frac{I_z - I_x}{I_y} pr \quad (7.18)$$

$$C_n = 0 \quad (7.19)$$

What do each of these results imply about a stable condition like the fully developed spin?

#### ● 7.3.3.1 Pitching Moment Balance.

By examining equation 7.18 in conjunction with the  $C_{m,b}$  versus  $\alpha$  curve for an aircraft it is at least possible to identify regions where a fully developed spin can occur.

First, the angle of attack must be above the stall angle of attack. This condition is obvious, since the definition of a spin demands  $\alpha > \alpha_s$ .

Second,  $C_{m,b}$  must be opposite in sign to the inertial term on the right hand side of equation 7.18. For an upright spin this requirement means that  $C_{m,b}$  must be negative. This fact is clear if one observes that  $I_z > I_x$  and that  $p$  and  $r$  are of the same sign in an upright spin (figure 7.12). In fact it is possible to express the rotation rate in a convenient form by slightly rearranging equation 7.18. Recall that

$$\frac{M_{\text{zero}}}{I_y} = \frac{V^2 C_{m,b}}{K_x^2 m}$$

Figure 7.12 also illustrates the fact that

$$p = \omega \cos \alpha \text{ and } r = \omega \sin \alpha$$

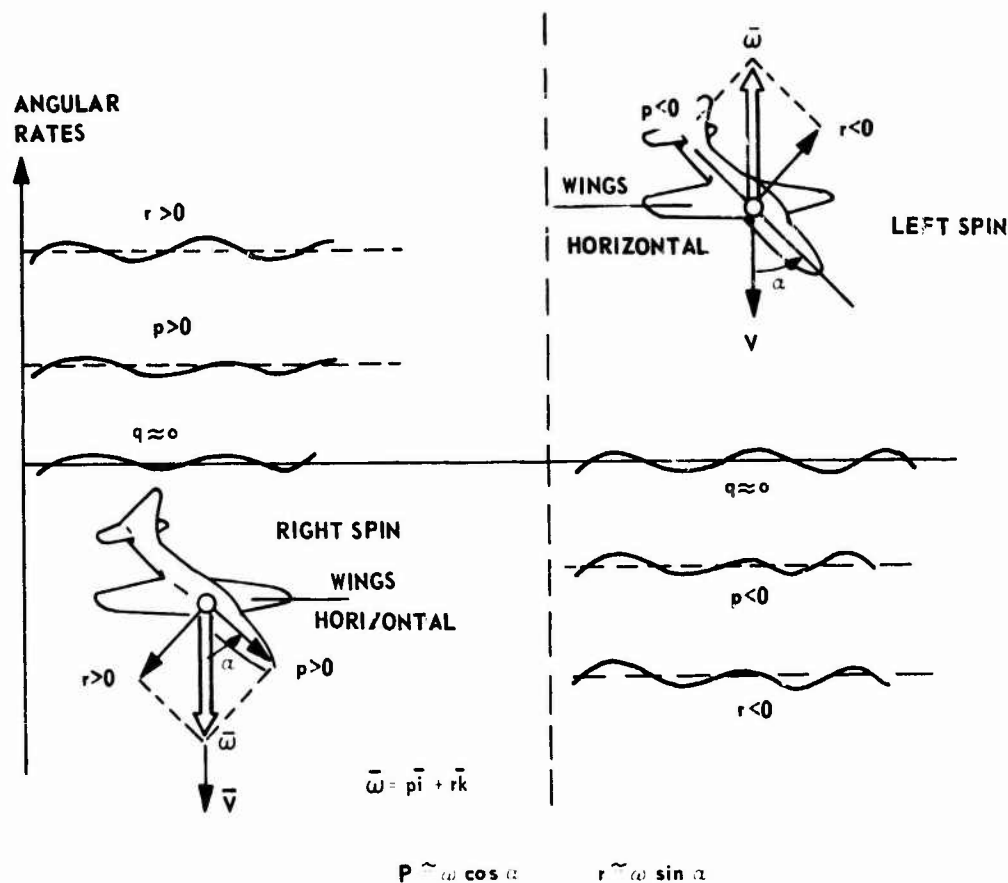


FIGURE 7.12 SPIN VECTOR COMPONENTS

Substituting into equation 7.18,

$$-\frac{M_{\text{aero}}}{I_y} = \frac{I_z - I_x}{I_y} \omega^2 \cos \alpha \sin \alpha$$

$$\omega^2 = \frac{-M_{\text{aero}}}{\frac{1}{2} (I_z - I_x) \sin 2\alpha} \quad (7.20)$$

Equation 7.20 suggests that the minimum rotation rate occurs near an  $\alpha$  of 45 degrees, although strong variations in  $M_{\text{aero}}$  may preclude this minimum. In fact, there is one additional prerequisite which must be satisfied before a fully developed spin can occur.

The slope of  $C_{m,b}$  versus  $\alpha$  must be negative or stabilizing and must be relatively constant. This is required simply because a positive

$\frac{dC_{m,b}}{d\alpha}$  represents a divergent situation and would therefore require a pitching acceleration,  $\dot{q} \neq 0$ . But this angular acceleration would violate the assumption of a constant  $\omega$  in a fully developed spin. Said another way, any disturbance in angle of attack would produce a  $\Delta C_{m,b}$  tending to

restore  $C_{m,b}$  to its initial value only so long as  $\frac{dC_{m,b}}{d\alpha} \neq 0$ . To summarize these constraints, consider figure 7.13. Aircraft B can enter a fully

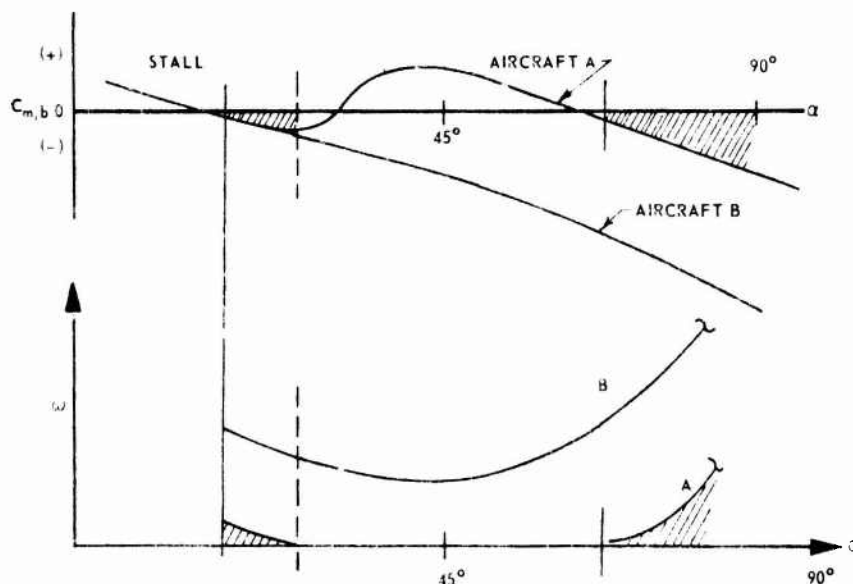


FIGURE 7.13 AERODYNAMIC PITCHING MOMENT PREREQUISITES

developed, upright spin at any AOA above  $\alpha_s$  insofar as the pitching moment equation is concerned because its  $C_{m,i}$  versus  $\alpha$  is always negative

and  $\frac{dC_{m,b}}{d\alpha}$  is always negative. However, aircraft A can meet the three

constraints imposed by the pitching equation only in the shaded areas. Of course, the pitching moment equation is not the sole criterion; the rolling and yawing moment equations must also be considered.

#### ●7.3.3.2 Rolling and Yawing Moment Balance.

Equations 7.17 and 7.19 suggest at least four other conditions which must be satisfied to have a fully developed spin occur. Although not specifically pointed out in paragraph 7.3.3.1 all the aerodynamic derivatives, even  $C_{m,b}$  are functions of both  $\alpha$ ,  $\beta$ , and the rotation rate  $\omega$  (reference 9, page 6). Having considered  $C_{m,b}$  as a function of  $\alpha$  alone, it is convenient to consider  $C_n$  and  $C_\ell$  as functions of  $\omega$  alone. There is little justification for this choice other than the fact the lateral-directional derivatives are more directly linked to rotation rate while the longitudinal derivative is more directly linked to angle of attack. But it is well to keep in mind that all these variables do affect  $C_{m,b}$ ,  $C_\ell$ , and  $C_n$ .

The conditions imposed by both  $C_n$  and  $C_\ell$  to allow a fully developed spin are that the derivatives must be equal to zero and the rate of change of the derivatives with respect to changes in  $\omega$  must be negative. The first of these conditions is explicitly stated by equations 7.17 and

7.19. But the second requirement ( $\frac{dC_\ell}{d\omega} < 0$  and  $\frac{dC_n}{d\omega} < 0$ ) stems from the

fact that a fully developed spin must be a stable condition. If an increase in  $\omega$  will produce an increased  $C_\ell$  or  $C_n$ , then any change in rotation rate will cause the autorotative moments to diverge away from the supposedly stable initial condition. Figure 7.14 illustrates this point.

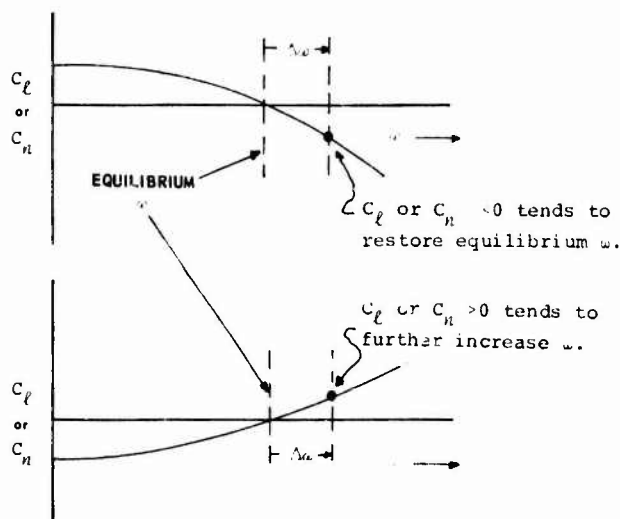


FIGURE 7.14 STABILIZING AND DESTABILIZING SLOPES FOR  $C_\ell$  AND  $C_n$  VERSUS  $\omega$

Obviously, these aerodynamic prerequisites must all be met for a fully developed spin to exist in a true equilibrium form. Of course, oscillatory spins may occur with some relaxation of one or more of these conditions. It is extremely rare to observe an ideal case which would precisely meet all these conditions in an actual spin. So, while exactly satisfying all these conditions is essential for a fully developed spin to actually exist, it is common to estimate spin parameters with less than perfect fulfillment of these prerequisites. An example of how such estimations are made will be considered next.

#### ●7.3.4 ESTIMATION OF SPIN CHARACTERISTICS

Reference 9, appendix B, describes in detail a method of estimating spin characteristics which was designed to estimate initial conditions for a computer study investigating possible steady state spin modes of the McDonnell F-3H Demon. Although this estimation method was only intended to help predict initial conditions for the numerical integration and thus save computer time, it serves as an excellent example of how model data and the aerodynamic prerequisites discussed in paragraph 7.3.3 can be combined to get a "first cut" at spin characteristics.

The aerodynamic data on which this example is based were measured by steadily rotating a model about an axis parallel to the relative wind in a wind tunnel. Hence, no oscillations in angular rates are taken into account. This limitation on the aerodynamic data is indicated by the subscript "r b" (rotation-balance tunnel measurements). In addition, the data are presented as a function of a nondimensional rotation rate,

$\frac{\omega_b}{2V}$ . To help simplify the estimation process and partly because the rolling moment data were not as "well-behaved" as the yawing moment data, the rolling moment data were ignored. However, all the other prerequisites of paragraph 7.3.3 were observed. The estimation method is outlined below and the interested student is referred to reference 9, page 18, for a fuller description and a numerical example.

##### ●7.3.4.1 Determining $C_{m,rb}$ From Aerodynamic Data.

Use the  $\frac{\omega_b}{2V}$  and  $\gamma$  for which  $C_{n,rb} = 0$  and  $\frac{dC_{n,rb}}{d(\frac{\omega_b}{2V})} = 0$  to determine

$C_{m,rb}$ . This amounts to using the model data to determine aerodynamic pitching moments for which the aerodynamic yawing moment is zero.

##### ●7.3.4.2 Calculating Inertial Pitching Moment.

Using a modified form of equation 7.20, and recognizing that the inertial pitching moment is the negative of the aerodynamic pitching moment on a fully developed spin,  $-C_{m,rb}$  is calculated.

$$\begin{aligned}\omega^2 &= \frac{-M_{aero}}{\frac{1}{2} (I_z - I_x) \sin 2 \alpha} \\ &= \frac{-C_{m,rb} \frac{1}{2} \rho v^2 S b}{\frac{1}{2} (I_z - I_x) \sin 2 \alpha}\end{aligned}$$

Solving for  $-C_{m,rb}$

$$-C_{m,rb} = \frac{I_z - I_x}{\rho S b} \left(\frac{\omega}{v}\right)^2 \sin 2 \alpha \quad (7.21)$$

#### ●7.3.4.3 Comparing Aerodynamic Pitching Moment and Inertial Pitching Moment.

Plot  $C_{m,rb}$  versus  $\alpha$  from the wind tunnel data (paragraph 7.3.4.1) and the results of equation 7.21 on the same plot, like figure 7.15.

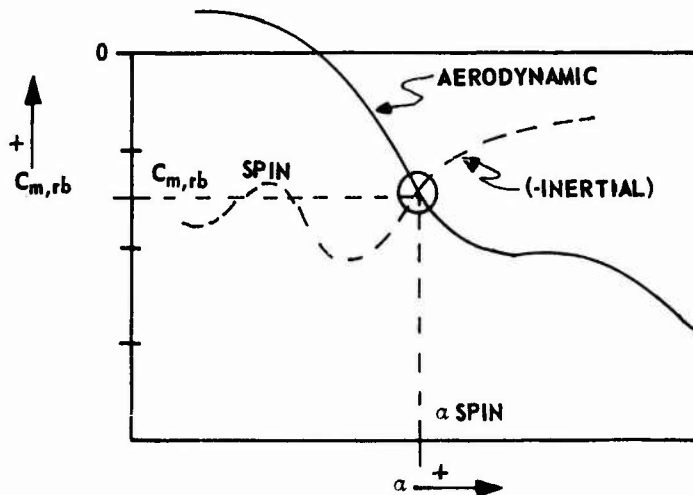


FIGURE 7.15 AERODYNAMIC PITCHING MOMENTS COMPARED TO INERTIAL PITCHING MOMENTS

The intersection of the two curves indicates a possible fully developed spin. From this plot the angle of attack of the potential spin is read directly and the value of  $C_{m,rb}$  is used to calculate the potential rotation rate.

#### ●7.3.4.4 Calculation of $\omega$ .

Rearranging equation 7.21,

$$\left(\frac{\omega}{V}\right)^2 = \frac{(-C_{m,rb})(\rho S b)}{(I_z - I_x) \sin 2\alpha} \quad (7.22)$$

the ratio  $\frac{\omega}{V}$  can be calculated. But equation 7.1 allows calculation of  $V$  if  $C_D$  is known. The model force measurements provide  $C_D$  and then

$$V^2 = \frac{W}{\frac{1}{2} \rho S C_D} \quad (7.23)$$

Then, of course,

$$\omega^2 = \frac{(-C_{m,rb})(\rho S b) W}{(I_z - I_x) (\sin 2\alpha) \frac{1}{2} \rho S C_D} \quad (7.24)$$

$$\omega^2 = \frac{-2C_{m,rb} b W}{C_D (I_z - I_x) \sin 2\alpha}$$

#### ●7.3.4.5 Results.

A typical set of results from the numerical integration of the six equations compared with the estimated parameters is given in table III below (extracted from reference 9, pages 26, 27).

Table III

TYPICAL COMPUTER RESULTS VERSUS ESTIMATION

Computer Results			Estimation		
$\alpha$ (deg)	$\omega$ (rad/sec)	V (ft/sec)	$\alpha$ (deg)	$\omega$ (rad/sec)	V (ft/sec)
36.0	1.88	294	38.2	1.90	285
37.0	1.92	372	45.1	1.83	327
Oscillated out of spin			48.2	1.89	453
51.8	2.18	619	50.5	2.18	620
80.0	4.72	494	70.0	3.50	515
36.5	2.80	380	37.4	2.69	365

### ●7.3.5 GYROSCOPIC INFLUENCES:

Only aerodynamic moments have been considered so far in expanding the applied external moments. Ordinarily the aerodynamic moments are the dominant ones, but gyroscopic influences of rotating masses can also be important. The NF-104, for example has virtually no aerodynamic moments at the top of its rocket-powered zoom profile. There is convincing evidence that gyroscopic moments from the engine dominate the equations of motion at these extreme altitudes (reference 10, page 13). The external applied moments should be generalized to include gyroscopic influences and other miscellaneous terms (anti-spin rockets, anti-spin chutes, etc.). The applied external moments become

$$G_x = L_{aero} + L_{gyro} + L_{other}$$

$$G_y = M_{aero} + M_{gyro} + M_{other}$$

$$G_z = N_{aero} + N_{gyro} + N_{other}$$

The next paragraph will consider a simplified expansion of the gyroscopic terms.

#### ●7.3.5.1 Gyroscope Theory.

By virtue of its rotation, a gyroscope tends to maintain its spin axis aligned with respect to inertial space. That is, unless an external torque is applied, the gyro spin axis will remain stationary with respect to the fixed stars. If a torque is applied about an axis which is perpendicular to the spin axis, the rotor turns about a third axis which is orthogonal to the other two axes. On removing this torque the rotation (precession) ceases - unlike an ordinary wheel on an axle which keeps on rotating after the torque impulse is removed.

These phenomena, all somewhat surprising when first encountered, are consequences of Newton's laws of motion. The precessional behavior represents obedience of the gyro to Newton's second law expressed in rotation form, which states that torque is equal to the time rate of change of angular momentum.

$$\bar{T} = \frac{d \bar{H}}{dt} \quad (7.25)$$

with  $\bar{T}$  = external torque applied to the gyroscope

$\bar{H}$  = angular momentum of the rotating mass

$$H = I \Omega$$

with  $I$  = moment of inertia of the rotating mass

$\Omega$  = angular velocity of the rotating mass

Equation 7.25 applies, like all Newton's laws, only in an inertial frame of reference. If it assumed that  $\vec{H}$  is to be expressed within a frame of

reference rotating at the precession rate of the gyroscope,  $\dot{\vec{H}}_{\text{inertial}} = \dot{\vec{H}}_{\text{rotating}} + \vec{\omega}_p \times \vec{H}$ . If the gyro spin rate is unchanged, then  $\dot{\vec{H}}$  measured in the rotating frame will be zero and equation 7.25 becomes

$$\vec{T} = \vec{\omega}_p \times \vec{H} \quad (7.26)$$

The direction of precession for a gyro when a torque is applied is given by equation 7.26. This direction is such that the gyro spin axis tends to align itself with the total angular momentum vector, which in this case is the vector sum of the angular momentum due to the spinning rotor and the angular momentum change due to the applied torque,  $\Delta \vec{H}$  as shown in figure 7.16. The law of precession is a reversible one. Just as a torque input results in an angular velocity output (precession), an angular velocity input results in a torque output along the corresponding axis.

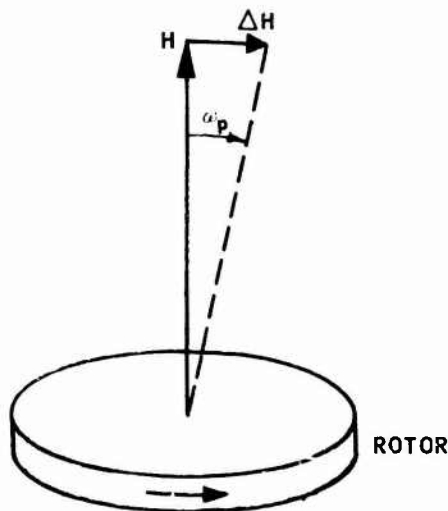


FIGURE 7.16 DIRECTION OF PRECESSION

Three gyro axes are significant in describing gyro operation; the torque axis, the spin axis and the precession axis. These are commonly referred to as input (torque), spin, and output (precession). The directions of these axes are shown in figure 7.17; they are such that the spin axis rotated into the input axis gives the output axis direction by the

right hand rule. The direction of rotational vectors such as spin, torque, and precession can be shown by means of the right hand rule. If the curve of the fingers of the closed right hand point in the direction of rotation, the thumb extended will point along the axis of rotation. For gyro work, it is convenient to let the thumb, forefinger, and middle finger represent the spin, torque, and precession axes respectively. Figure 7.18 illustrates this handy memory device.

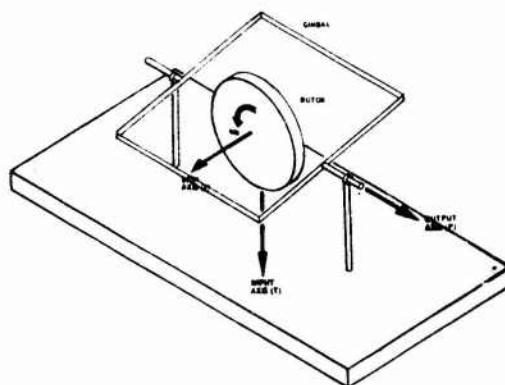


FIGURE 7.17 GYRO AXES

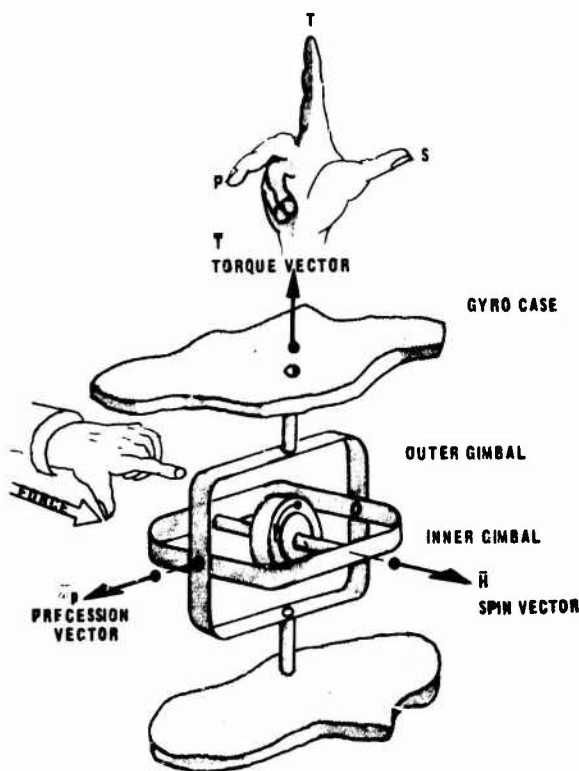


FIGURE 7.18 SPIN, TORQUE AND PRECESSION VECTORS

### ●7.3.5.2 Engine Gyroscopic Moments.

In figure 7.19 consider the rotating mass of the engine as a gyroscope and analyze the external torque applied to the engine by the engine mounts of an aircraft in a spin. Then the total angular velocity of the rotating mass is the vector sum of  $\bar{\omega}_E + \bar{\omega}$ , with  $\bar{\omega}_E$  being the engine rpm (assumed constant) and  $\bar{\omega}$  being the aircraft's spin rotation rate.

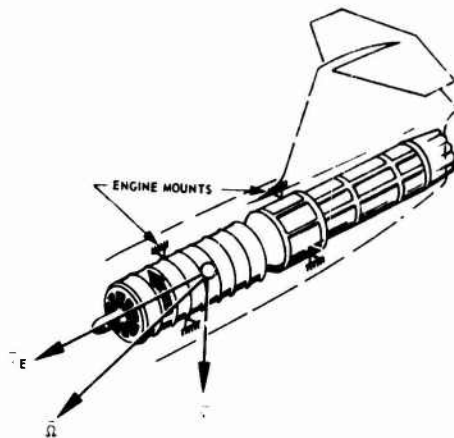


FIGURE 7.19 ANGULAR VELOCITIES OF THE ENGINE'S ROTATING MASS

$$\bar{\omega} = \bar{\omega}_E + \bar{\omega}$$

But  $\bar{\omega} = \bar{\omega}_E + \bar{\omega}$

$$\bar{\omega} = \bar{\omega}_E$$

If one also assumes that the rotational axis of the engine is parallel to the x-axis,

$$\bar{\omega} = \omega_E \hat{i}$$

Then the angular momentum of the engine is

$$\bar{H}_E = I_E \omega_E \hat{i}$$

with  $I_E$  = moment of inertia of the engine about the x-axis.

Considering figure 7.19 again and applying equation 7.26, the external torque applied to the engine must be the precession rate of the aircraft ( $\bar{\omega}$ ) crossed into the engine's angular momentum.

$$\bar{T} = \bar{\omega} \times \bar{H}_E$$

But the moment applied by the engine through the engine mounts to the spinning aircraft is equal but opposite in sign (Newton's Third Law).

$$\bar{G}_{\text{gyro}} = -\bar{\omega} \times \bar{H}_E$$

$$\begin{bmatrix} L_{\text{gyro}} \\ M_{\text{gyro}} \\ N_{\text{gyro}} \end{bmatrix} = - \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ I_E \omega_E & 0 & 0 \end{bmatrix}$$

$$L_{\text{gyro}} = 0 \quad (7.27)$$

$$M_{\text{gyro}} = -I_E \omega_E r \quad (7.28)$$

$$N_{\text{gyro}} = I_E \omega_E q \quad (7.29)$$

Then equations 7.11, 7.12, and 7.13 can be expanded to

AERO	INERTIAL COUPLING (sometimes called gyrodynamic term)	GYROSCOPIC TERM (an engine effect)	MISC (rockets, spin chutes, etc.)	
$\dot{p} = \frac{L_{\text{aero}}}{I_x} +$	$\frac{I_y - I_z}{I_x} q r$	$+ \frac{L_{\text{gyro}}}{I_x}$	$+ \frac{L_{\text{other}}}{I_x}$	(7.30)

$\dot{q} = \frac{M_{\text{aero}}}{I_y} +$	$\frac{I_z - I_x}{I_y} p r$	$+ \frac{M_{\text{gyro}}}{I_y}$	$+ \frac{M_{\text{other}}}{I_y}$	(7.31)
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$\dot{r} = \frac{N_{\text{aero}}}{I_z} +$	$\frac{I_x - I_y}{I_z} p q$	$+ \frac{N_{\text{gyro}}}{I_z}$	$+ \frac{N_{\text{other}}}{I_z}$	(7.32)
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Equation 7.20 becomes

$$\omega^2 = \frac{-M_{\text{aero}} + I_E \omega_E r}{\frac{1}{2} (I_z - I_x) \sin 2 \alpha} \quad (7.33)$$

Equation 7.33 shows that the effect of the engine gyroscopic moment is to shift the curves of figure 7.13 as shown below. An engine that rotates in a clockwise direction (as viewed from the tailpipe) will cause an aircraft to spin faster in a right spin and slower in a left spin. Generally speaking, however, this engine gyroscopic moment is negligible in comparison to the other external moments.

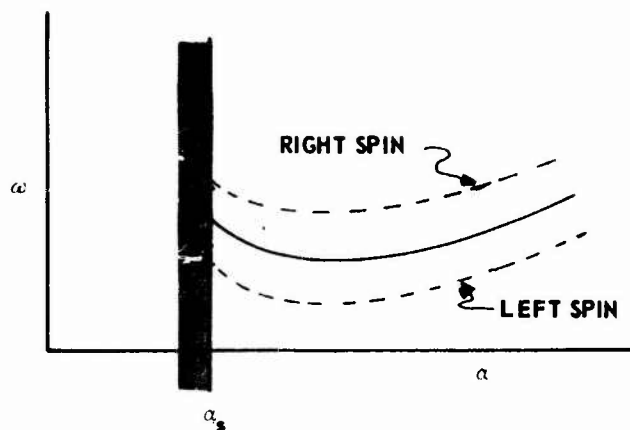


FIGURE 7.20 EFFECT OF  $M_{gyro}$  ON SPIN ROTATION RATE

### ●7.3.6 SPIN CHARACTERISTICS OF FUSELAGE-LOADED AIRCRAFT:

It is appropriate to consider briefly some of the spin characteristics peculiar to modern high performance aircraft in which the mass is generally concentrated within the fuselage ( $I_y$  larger than  $I_x$  and almost as large as  $I_z$ ). It can be shown that a system which has no external moments or forces tends to rotate about its largest principal axis, which in the case of an aircraft, is the Z axis. In an actual spinning aircraft, the external moments are not zero and thus the aircraft spins about some intermediate axis. For the idealized spin thus far considered, the pitching moment equation leads one to the observation that fuselage-loaded aircraft will probably spin flatter than their wing-loaded counterparts.

#### ●7.3.6.1 Fuselage-Loaded Aircraft Tend to Spin Flatter Than Wing-Loaded Aircraft.

For a fully developed spin

$$G_y = -pr(I_z - I_x) \quad (7.34)$$

In an aircraft,  $(I_z - I_x)$  can never be zero. Hence, if  $G_y = 0$  then  $p$  must be zero, in which case  $\bar{\omega} = r\bar{k}$  and the spin is flat ( $\bar{\omega} = p\bar{i}$  is excluded by the definition of a spin). If the spin is not flat, then both  $p$  and  $r$  exist and, in an upright spin, have the same algebraic sign. Because  $(I_z - I_x)$  is always positive, examination of equation 7.34 shows that  $G_y$  must always be negative (or zero) for an upright spin.

The smaller the pitch attitude ( $\theta$  in figure 7.21) the flatter the spin, and  $\theta$  can be defined as  $\sin^{-1} \frac{p}{\omega}$  for the spin depicted in figure 7.21.  $\theta$  varies with the relative magnitude of  $(I_z$  and  $I_x)$ , as can readily be seen by rearranging equation 7.34.

$$p = \frac{|G_y|}{r(I_z - I_x)}$$

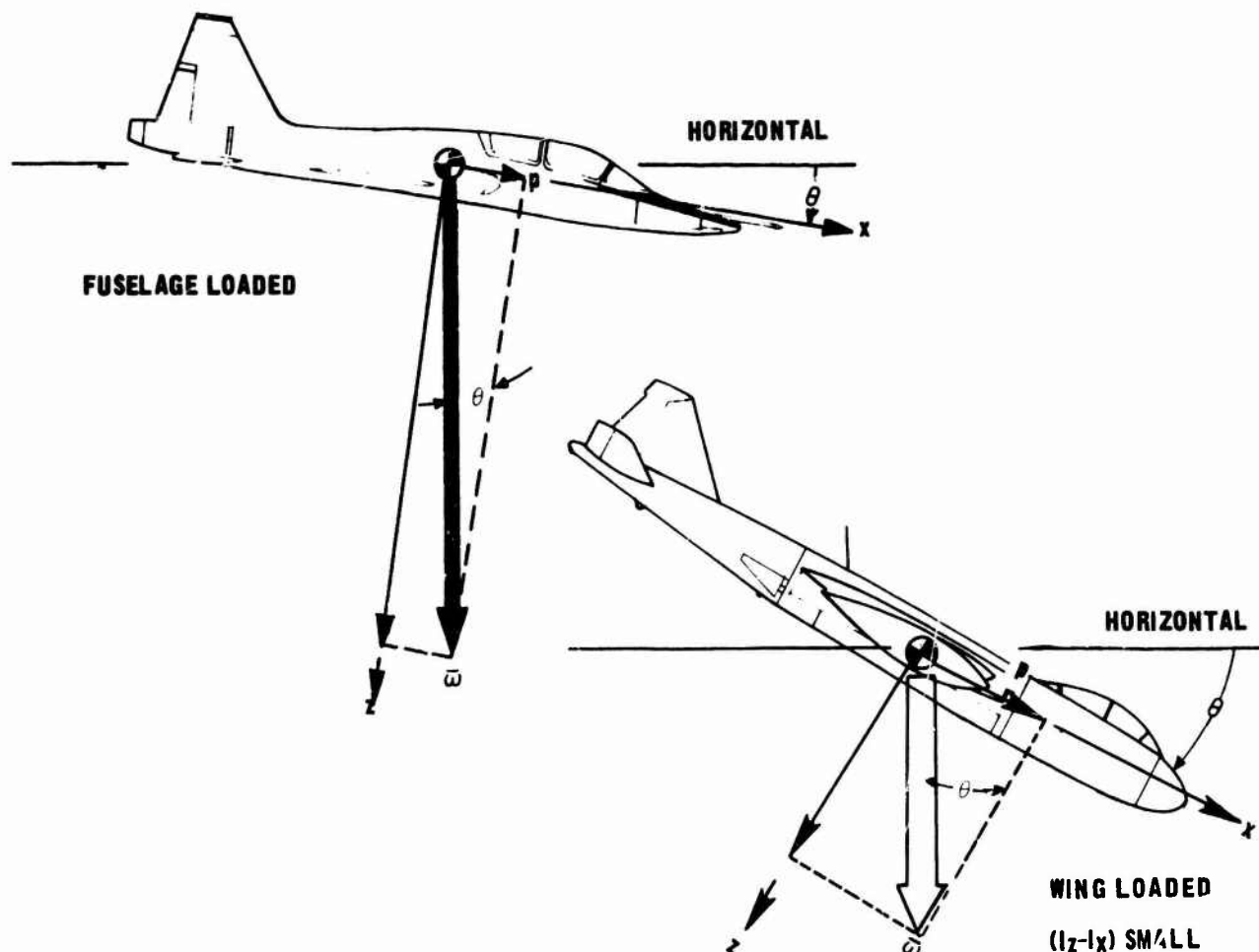


FIGURE 7.21 EFFECT OF MAGNITUDES OF  $I_z$  AND  $I_x$  ON SPIN ATTITUDE

Since  $p$  becomes smaller as  $(I_z - I_x)$  increases, it is clear that fuselage-loaded aircraft tend to spin flatter than wing-loaded aircraft. But what about the effect of increasing  $I_y$  upon the roll equation?

#### ●7.3.6.2 Fuselage-Loaded Aircraft Tend to Exhibit More Oscillations.

On aircraft where  $I_y$  is approximately equal to  $I_z$  in magnitude, the fully developed spin is more likely to be oscillatory. In the limit, if  $I_y = I_z$ , the reference spin could be wing down, since any axis in the  $YZ$  plane would be a maximum principal axis. Although these facts suggest that the bank angle is easily disturbed and that a developed spin often occurs with the bank angle not zero, a restoring tendency does exist which leads to periodic oscillations in bank angle. Consider again the rolling moment equation,

$$G_x = \dot{p} I_x + q r (I_z - I_y) \quad (7.35)$$

If an "o" subscript is used to represent the reference or steady-state conditions,

$$G_{x_o} = \dot{p}_o I_x + q_o r_o (I_z - I_y)$$

If instantaneous values are represented by equation 7.35, the change in external moments due to the perturbations of the angular acceleration and angular velocities is

$$(G_x - G_{x_o}) = (\dot{p} - \dot{p}_o) I_x + (q r - q_o r_o) (I_z - I_y)$$

Assuming perturbations in roll with not significantly change  $r_o$ ,  $r \approx r_o$  and

$$\begin{aligned} \Delta G_x &= \Delta \dot{p} I_x + \Delta q (I_z - I_y) r_o \\ \Delta \dot{p} &= \frac{\Delta G_x}{I_x} - \Delta q \frac{I_z - I_y}{I_x} r_o \end{aligned} \quad (7.36)$$

The second term on the right side of equation 7.36 serves to damp oscillations in that it reduces the ability of perturbations in rolling moment ( $\Delta G_x$ ) to produce perturbations in roll acceleration ( $\Delta \dot{p}$ ). For fuselage-loaded aircraft, in which  $(I_z - I_y)$  is small, the damping is much reduced. Thus, any perturbations in the motion tend to persist longer in fuselage-loaded aircraft than they do in wing-loaded aircraft.

### ● 7.3.7 SIDESLIP:

It is beyond the scope of this course to deal with the effects of sideslip in any detail. However, it is noteworthy that sideslip need not be zero in a developed spin; in fact it usually is not. Reference 7, page 535, shows that sideslip in a spin arises from two sources: wing tilt with respect to the horizontal ( $\phi$ ) and the inclination of the flight path to the vertical ( $\eta$ ).

$$\beta \approx \phi - \eta \quad (7.37)$$

If then, one considers a spin with a helical flight path as opposed to a vertical flight path, the inclination of the flight path to the vertical is positive and equal to the helix angle. Then, in order to maintain zero sideslip, the retreating wing must be inclined downwards by an amount equal to the helix angle in order to have zero sideslip. However, it is quite common to have fully developed spins (with the spin axis vertical, not the flight path) with varying amounts of sideslip. Sideslip on a stalled wing will generally increase the lift on the wing toward which the sideslip occurs and reduce the lift on the opposite wing. It is easy to understand that a small amount of sideslip can produce a large rolling

moment and thereby significantly alter the balance of rolling moments. These qualitative comments are quite cursory and the inquisitive student may wish to pursue these effects further. Reference 7 offers an expanded discussion, but to adequately discuss sideslip effects in any detail one must consider all three moment equations and their coupling effects. The consideration of sideslip leads to the general conclusion that the rolling couple can be balanced over a wide range of angles of attack and spin rotation rates.

#### ■ 7.4 INVERTED SPINS

Since PSG's are definitely uncontrolled aircraft motions, there is absolutely no guarantee that all spins will be of an upright variety, as has so far been assumed. The test pilot particularly (and operational pilots as well) will continue to experience inverted spins and PSG's which may be mainly inverted aircraft motions. As reference 11, page 1, points out,

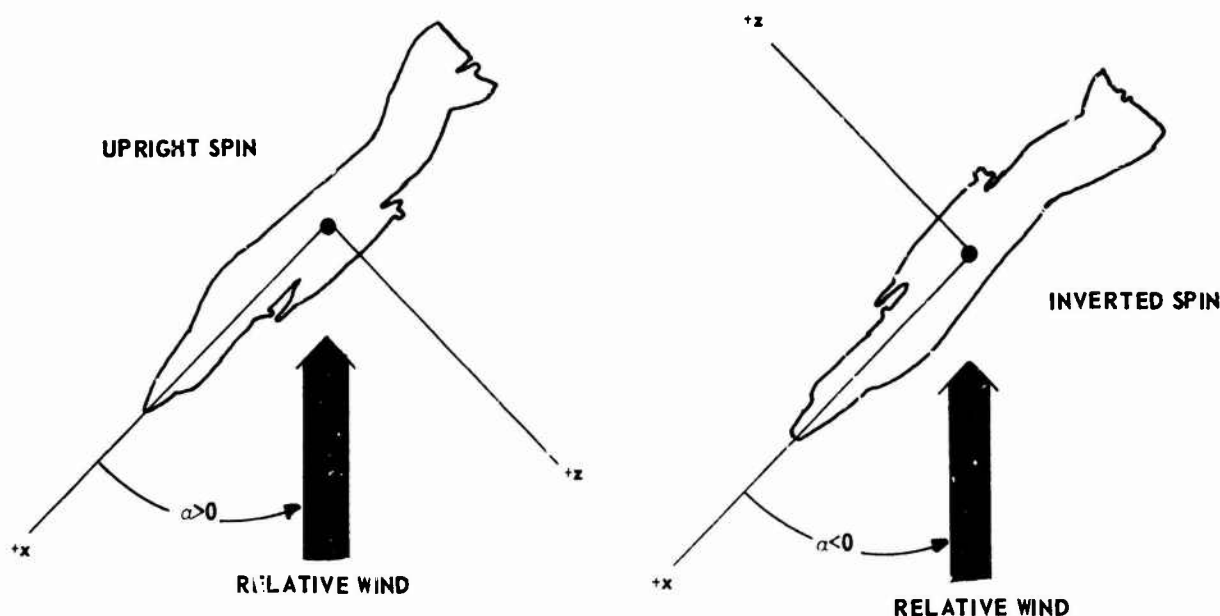
"...inverted spins cannot be prevented by handbook entries that 'the airplane resists inverted spins'."

It is, therefore, essential that the test pilot have some appreciation of the nature of the inverted PSG/spin. As usual, the analytical emphasis will necessarily be restricted to the fully developed spin, but the qualitative comments which follow also apply in a general way to other types of post-stall motion.

The most common pilot reaction to an inverted post-stall maneuver is, "I have no idea what happened! The cockpit was full of surprise, dirt, and confusion." Why? First, negative  $g$  flight is disconcerting in and of itself, particularly when it is entered inadvertently. But even experienced test pilots can be upset and their powers of observation reduced in an anticipated inverted spin. This disorientation usually takes one of two forms: (1) inability to distinguish whether the motion is inverted or upright or (2) inability to determine the direction of the spin. Each of these problems will be considered separately.

##### ● 7.4.1 ANGLE OF ATTACK IN AN INVERTED SPIN

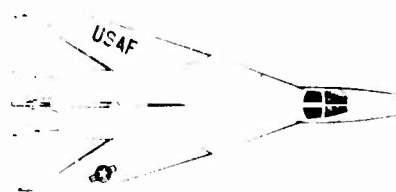
The angle of attack in an inverted spin is always negative (figure 7.22). It might appear that it would be easy to determine the difference in an upright or inverted spin; if the pilot is "hanging in the straps," it is an inverted spin. Such an "analysis" is accurate in some spin modes (the Hawker Hunter has an easily recognized smooth, flat mode such as this); however if the motion is highly oscillatory, not fully developed, or a PSG, the pilot's tactile senses are just not good enough. If the aircraft has an angle of attack indicator, this is probably the most reliable means of determining whether the maneuver is erect or inverted. Lacking an angle of attack system, the pilot must rely on the accelerometer or his sensory cues, neither of which are easy to interpret. But what about determining spin direction?



**FIGURE 7.22 ANGLE OF ATTACK IN AN INVERTED SPIN**

#### ● 7.4.2 ROLL AND YAW DIRECTIONS IN AN INVERTED SPIN

Consider two identical aircraft, one in an upright spin and the other in an inverted spin as shown in figure 7.23. Notice that the spin direction in either an upright or an inverted spin is determined by the sense of the yaw rate. Notice also that in an inverted spin the sense of the roll rate is always opposite to that of the yaw rate. It is common for pilots to mistakenly take the direction of roll as the spin direction. The chances of making this error are considerably enhanced during a PSG or the incipient phase of the spin when oscillations are extreme. In steep inverted spins ( $|\alpha|$  nearly equals  $|u_s|$ ) the rolling motion is the largest rotation rate and further adds to the confusion. However, there is a reliable cockpit instrument, the turn needle, which always indicates the direction of yaw. With such confusion possible, what about the previously obtained equations of motion? Is it necessary to modify them for the inverted spin?



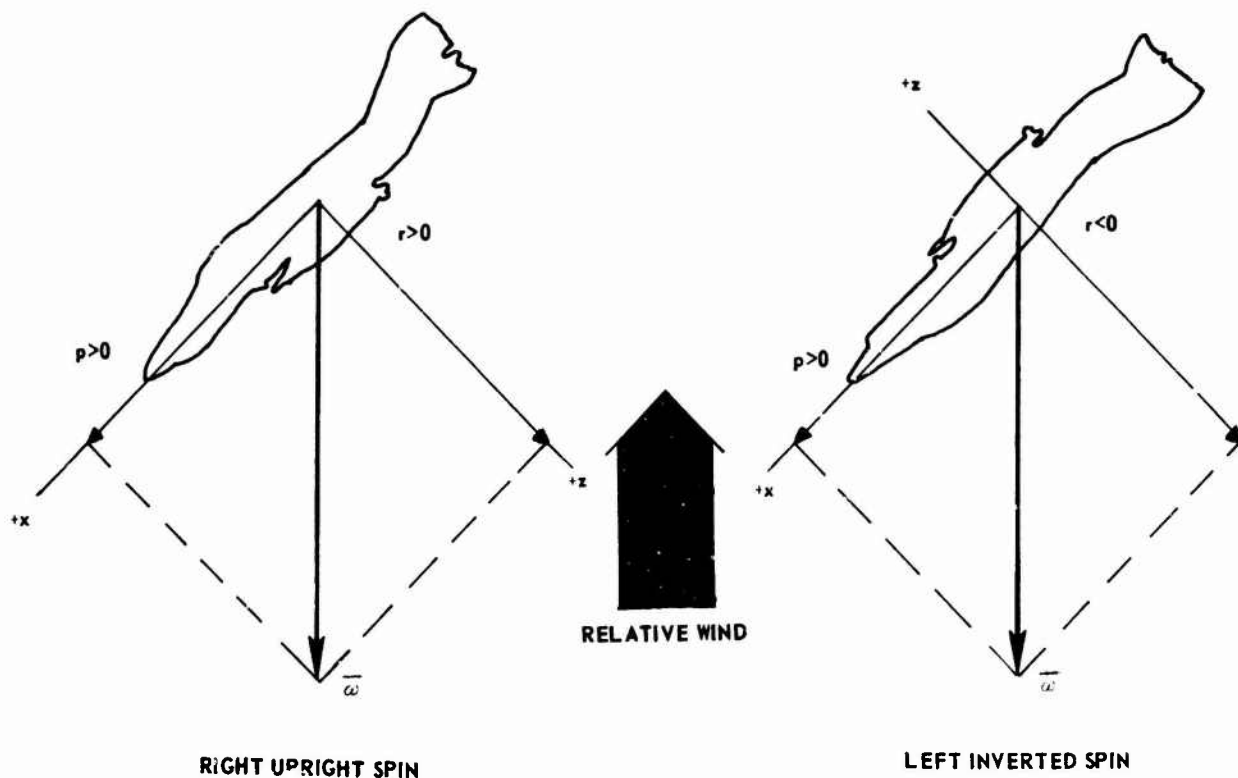


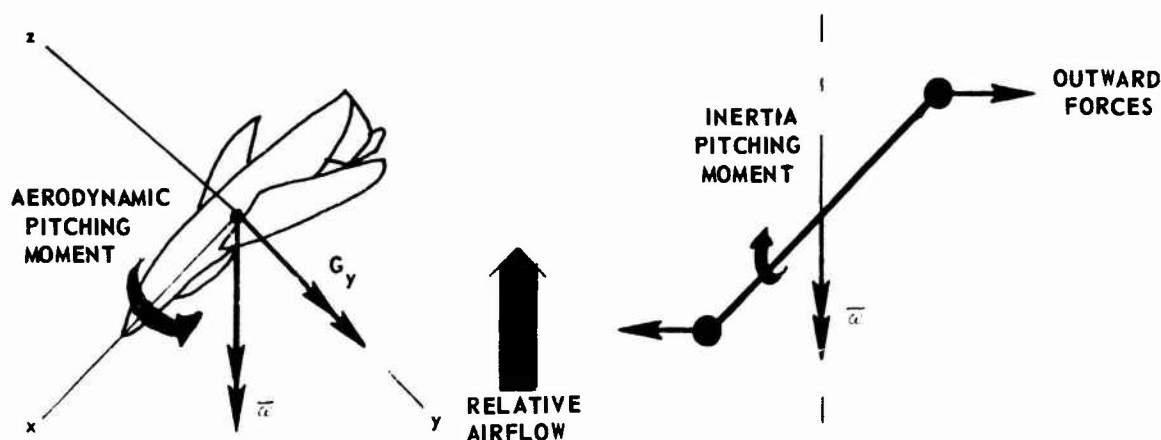
FIGURE 7.23 ROLL AND YAW RATES IN AN INVERTED SPIN

#### ●7.4.3 APPLICABILITY OF EQUATIONS OF MOTION:

All the equations previously described are directly applicable to the inverted spin. Of course, the differences in sign for angle of attack and the dearth of aerodynamic data collected at negative angle of attack pose a significant practical problem in trying to do detailed analyses of the inverted spin. But for the qualitative purposes of this course, the equations of motion are usable. However, it is instructive to note the difference in the sense of the pitching moments between an upright and an inverted spin. Recall that in an upright spin the applied external pitching moment (dominated by the aerodynamic pitching moment) had to be negative to balance the inertia couple, as equation 7.34 for a fully developed spin shows.

$$G_y = -pr(I_z - I_x) \quad (7.34)$$

But when  $p$  and  $r$  are of opposite, as in the inverted spin, the applied external moment must be positive. This fact is illustrated in figure 7.24, where the mass of the aircraft is represented as a rotating dumb-bell.



**FIGURE 7.24 PITCHING MOMENTS IN AN INVERTED SPIN**

It is apparent that in the inverted spin the external pitching moment is positive; that is, expressed as a vector, it lies along the positive y-axis. As a final point, the recovery from PSG's/spins, both erect and inverted, must be examined in some detail.

## ■ 7.5 RECOVERY

Obtaining developed spins today is generally difficult, but when obtained, the factors that make it difficult to obtain this type of spin may also make it difficult to recover from the spin. Current and future aircraft designs may be compromised too much for their intended uses to provide adequate aerodynamic control for termination of the developed spin; also, there is a problem of pilot disorientation associated with developed spins. As a result, the PSG and the incipient phase of the spin must be given more attention than they have received in the past, and preventing the developed spin through good design and/or proper control techniques has become a primary consideration.

Current aircraft have weights which are appreciably larger and have moments of inertia about the Y and Z axes which may be ten times as large as those of World War II aircraft. With the resulting high angular momentum, it is difficult for a spin to be terminated as effectively as a spin in earlier airplanes by aerodynamic controls which are generally of similar size. Furthermore, controls which are effective in normal flight may be inadequate for recovery from the spin unless sufficient consideration has been given to this problem in the design phase.

### ● 7.5.1 TERMINOLOGY

The recovery phase terminology was purposely omitted from paragraph 7.1.4 for inclusion here. Referring to figure 7.1, the whole of the recovery phase begins when the pilot initiates recovery controls and ends when the aircraft is in straight flight; however, there are several terms used to differentiate between the subparts of this phase.

#### ● 7.5.1.1 Recovery.

Recovery is defined as the transitional event from out-of-control conditions to controlled flight. In more usable terms, this period of time normally is counted from the time the pilot initiates recovery controls and that point at which the angle of attack is below  $\alpha_S$  and no significant uncommanded angular motions remain. The key phrase in this expanded definition is "angle of attack below  $\alpha_S$ ;" once this objective is attained the aircraft can be brought back under control provided there are sufficient altitude and airspeed margins to maneuver out of whatever unusual attitude ensues.

#### ● 7.5.1.2 Dive Pullout and Total Recovery Altitude.

The dive pullout is the transition from the termination of recovery to level flight. Total recovery altitude is the sum of the altitude losses during the recovery and dive pullout. Notice that reference 3, paragraph 3.4.2.2.2, specifies altitude loss during recovery - not total recovery altitude.

#### ● 7.5.2 ALTERATION OF AERODYNAMIC MOMENTS:

The balanced condition of the developed spin must be disturbed in order to effect a recovery, and prolonged angular accelerations in the proper direction are needed. Several methods for obtaining these accelerations are available but not all are predictable. Also, the accompanying effects of some methods are adverse or potentially hazardous. The general methods available for generating anti-spin moments are presented with the applicable terms of the general equations given below. Alteration of the aerodynamic moments ( $C_l$ ,  $C_{m,b}$ , and  $C_n$ ) through the use of flight controls is the conventional means of spin recovery; seldom are configuration changes presently used to accomplish spin recovery. The all-important question is "How should the flight controls be used to recover from a PSG or a spin?"

<p>1. Modify aerodynamic moments</p> <p>a. With flight controls</p> <p>b. Configuration changes (gear, flaps, strakes)</p>	<p>2. Reposition the aircraft attitude on the spin axis</p>
$\dot{p} = \frac{v^2}{2 \cdot K_x} \left[ C_l + \frac{I_y - I_z}{I_x} q r + \frac{L_{other}}{I_x} \right]$	$\dot{q} = \frac{v^2}{2 \cdot K_y} \left[ C_{m,b} + \frac{I_z - I_x}{I_y} p r - \frac{I_E \beta_E}{I_y} r + \frac{M_{other}}{I_y} \right]$
$\dot{r} = \frac{v^2}{2 \cdot K_z} \left[ C_n + \frac{I_x - I_y}{I_z} p q + \frac{I_E \beta_E}{I_z} q + \frac{N_{other}}{I_z} \right]$	<p>3. Variations in Engine Power</p> <p>4. Spin chutes Spin Rockets</p>

#### ●7.5.2.1 Use of Longitudinal Control.

The longitudinal control surface can only be effective if it can drive the angle of attack below  $\alpha_S$ . Rarely is the elevator capable of producing this much change in pitching moment in a fully developed spin, but its use during a PSG or the incipient phase of a spin may well reduce angle of attack sufficiently. However, forward stick during a fully developed upright spin will merely cause many spin modes to progress to a higher rotation rate, which is also usually flatter. Model tests and computer studies should thoroughly investigate this control movement before it is recommended to the test pilot. Then a thorough flight test program must be conducted to confirm these predictions before such a recommendation is passed on to operational users.

#### ●7.5.2.2 Use of Rudder.

Considering only the alteration of  $C_l$ ,  $C_{m,b}$ , or  $C_n$  by deflection of the appropriate control surfaces, the use of rudder to change  $C_n$  has proven to be the most effective in recovering from a developed spin. Rudder deflection, if the rudder is not blanked out, produces a reduction in yaw rate which persists. The reduction in yaw rate reduces the inertia pitching couple and the angle of attack consequently decreases. Once the rotation rate has been reduced sufficiently, the longitudinal control can be used to reduce angle of attack below  $\alpha_S$ .

The student will notice that the use of ailerons to produce an anti-spin rolling moment has not been discussed. Generally, in stalled flight the ailerons are not effective in producing moments of any significance, though they can still be the primary anti-spin control by causing a small change in bank angle and thereby reorienting the aircraft attitude on the spin axis so that the inertial terms operate to cause recovery.

#### ●7.5.3 USE OF INERTIAL MOMENTS:

Examination of the inertial terms listed below reveals that the relative magnitudes of  $I_x$  and  $I_y$  determine how the ailerons should be used to reorient the aircraft attitude.

$$\dot{p} = \dots + \frac{I_y - I_z}{I_x} pr \dots$$

$$\dot{q} = \dots + \frac{I_z - I_x}{I_y} pq \dots$$

$$\dot{r} = \dots + \frac{I_x - I_y}{I_z} rp \dots$$

For a fuselage-loaded aircraft the pitch rate must be positive in an upright spin to develop anti-spin yawing and rolling accelerations from these terms. In both cases aileron applied in the direction of the spin causes the aircraft body axes to tilt so as to produce a positive component of  $\dot{p}$  along the y-axis (see figure 7.25). Another way to help achieve a positive pitch rate is to hold aft stick until the rotation rate

begins to drop. This procedure is common in some fuselage-loaded aircraft, although it is unacceptable in others (F-104) for example). However, the most important factor is the relative sizes of  $I_x$  and  $I_y$ . Considering that  $\frac{I_x - I_y}{I_z}$  is approximately six times greater for the F-104 than for the T-28, it is little wonder that aileron is a more important spin recovery control in the F-104 than is the rudder.

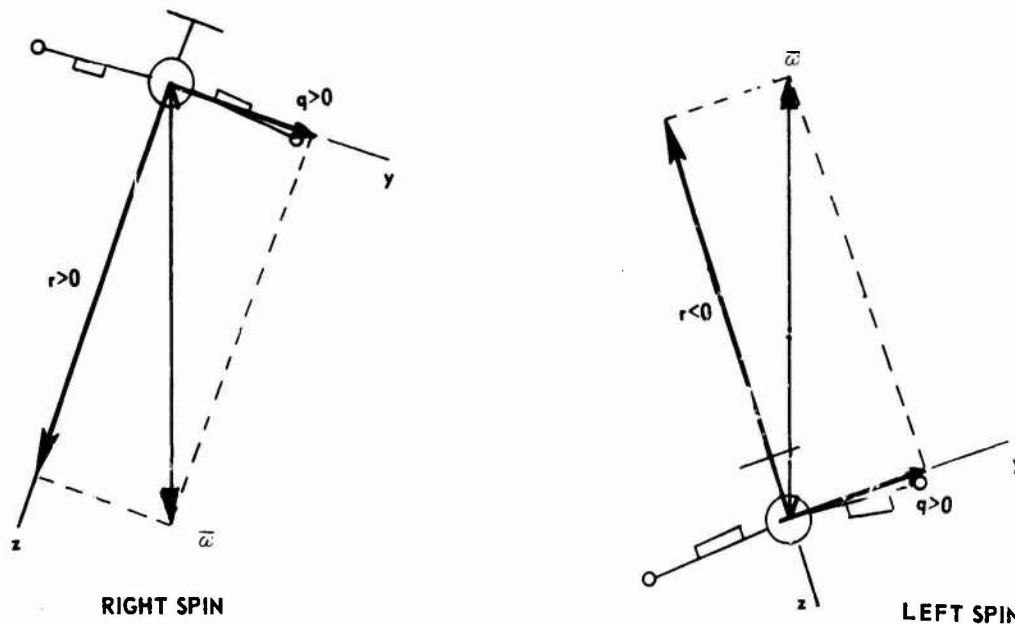


FIGURE 7.25 AILERON WITH RECOVERY PROCEDURE

A similar analysis shows that aileron against the upright spin in a wing-loaded aircraft will produce an anti-spin yaw acceleration, but a pro-spin roll acceleration. Since wing-loaded aircraft generally spin more nose low than fuselage-loaded aircraft (with  $p \approx r$ ), and since they generally are recoverable with rudder and elevator, aileron-against recovery procedures are rarely recommended.

#### ●7.5.4 OTHER RECOVERY MEANS:

The other two terms which can produce anti-spin accelerations include engine gyroscopic terms and emergency recovery devices.

##### ●7.5.4.1 Variations in Engine Power.

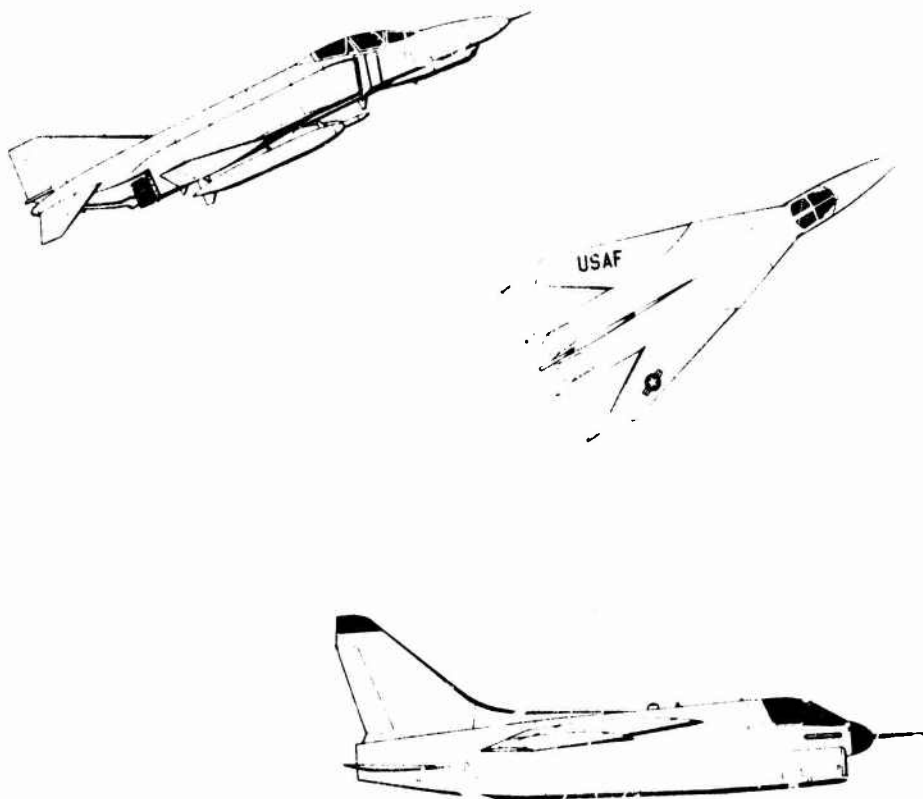
The gyroscopic terms are usually so small that they have little effect on recovery characteristics. Furthermore, jet engines often flame out during PSG or spin motions, particularly if the throttle is not at idle. So, although there are potential pitch and yaw accelerations available from the gyroscopic terms, NASA experience indicates that changes in engine power are generally detrimental to recovery.

#### 67.5.4.2 Emergency Recovery Devices.

Emergency recovery devices may take many forms - anti-spin parachutes attached to the aft fuselage, anti-spin parachutes attached to the wing tip, anti-spin rockets, strakes, etc. The design of such devices is a complex subject worthy of careful engineering in its own right. Certainly such design considerations are not the concern of test pilots; but the reliability of the device, its attachments, and its jettison mechanism are of vital concern to him. He is also likely to be concerned with tests to validate this reliability.

#### 67.5.5 RECOVERY FROM INVERTED SPINS:

Recovery from inverted spins is generally easier than recovery from upright spins, particularly if the rudder is in undisturbed airflow. In fact many aircraft will recover from an inverted spin as soon as the controls are neutralized. In any case rudder opposite to the turn needle may be recommended, often in conjunction with aft stick. Some fuselage-loaded T-tailed aircraft may require anti-spin aileron. An analysis similar to that in paragraph 7.5.3 shows that in an inverted spin aileron against the spin is the correct anti-spin control for a fuselage-loaded aircraft.



## ROLL COUPLING

## 8.1 INTRODUCTION

Divergencies experienced during rolling maneuvers have frequently been referred to as "Inertia Coupling." This leads to a misconception of the problems involved. The divergence experienced during rolling maneuvers is complex because it involves not only inertia properties, but aerodynamic ones as well. It is the intention of this chapter to offer a physical explanation of the more important causes of roll coupling and to also introduce a brief mathematical tool to aid in predicting roll coupling divergencies.

Coupling results when a disturbance about one aircraft axis causes a disturbance about another axis. An example of uncoupled motion is the disturbance created by an elevator deflection. The resulting motion is restricted to pitching motion and no disturbance occurs in yaw or roll. An example of coupled motion is the disturbance created by a rudder deflection. The ensuing motion will be some combination of both yawing and rolling motion. Although all lateral-directional motions are coupled, the only motion that ever results in coupling problems large enough to threaten the structural integrity of the aircraft is coupling as a result of rolling motion. Thus our study of "roll coupling."

Although there are numerous contributions to the roll coupling characteristics of an aircraft, aeroelastic effects, etc., this chapter will only consider three: (1) inertia coupling, (2) the  $I_{xz}$  parameter, (3) aerodynamic coupling.

These effects occur simultaneously in a very complicated fashion. Therefore, the resulting aircraft motion cannot be predicted by analyzing these effects separately. The complicated interrelationship of these parameters can best be seen by analyzing the aircraft equations of motion.

$$\text{Roll } \frac{\sum L}{I_x} = \dot{p} + qr \left( \frac{I_z - I_y}{I_x} \right) - (\dot{r} + qp) \frac{I_{xz}}{I_x} \quad (8.1)$$

$$\text{Pitch } \frac{\sum M}{I_y} = \dot{q} + pr \left( \frac{I_x - I_z}{I_y} \right) + (p^2 - r^2) \frac{I_{xz}}{I_y} \quad (8.2)$$

$$\text{Yaw } \frac{\sum N}{I_z} = \dot{r} + pq \left( \frac{I_y - I_x}{I_z} \right) + (qr - \dot{p}) \frac{I_{xz}}{I_z} \quad (8.3)$$

$$\text{Drag } \frac{\sum F_x}{m} = \dot{u} + qw - rv \quad (8.4)$$

$$\text{Lift } \frac{\sum F_z}{m} = \dot{w} + pv - qu \quad (8.5)$$

$$\text{Side } \frac{\sum F_y}{m} = \dot{v} + ru - pw \quad (8.6)$$

This analysis will be based on equations 8.1 - 8.3. Equations 8.4 - 8.6 are not important in an analysis of roll coupling. Consider equations 8.1 - 8.3. In each case,

the first term in the equations represents the aerodynamic contribution, the second term the inertial contribution, and the third term the  $I_{xz}$  parameter. It can be seen that the relationships involved could never occur singularly and that they actually occur in conjunction with one another to either become additive and aggravating or opposing and thus alleviate the tendency to diverge.

"Divergence" in roll coupling is manifested by a departure from the intended flight path that will result in either loss of control or structural failure. As defined, this "divergence" is what we are concerned with in roll coupling. Smaller roll coupling effects that do not result in divergence will not be considered. It should be noted that divergence about any one axis will be closely followed by divergence about the others.

In attempting to explain the subject of roll coupling, this chapter will first explain the physical aspects of inertia coupling, aerodynamic coupling, and the  $I_{xz}$  parameter. No attempt will be made to physically analyze these combined motions. Next, a mathematical model for roll coupling will be developed that will permit determination of the approximate roll rate that will drive an aircraft to divergence.

## 8.2 INERTIAL COUPLING

The problem of inertial coupling did not manifest itself until the introduction of the century series aircraft. As the modern fighter plane evolved from the conventional fighter, such as the F-51 and F-47, through the first jet fighter, the F-80, and then to the F-100 and other century series aircraft, there was a slow but steady change in the weight distribution. During this evolution, more and more weight became con-

centrated in the fuselage as the aircraft's wings became thinner and shorter. This shift of weight caused relations between the moments of inertia to change. As more weight is concentrated along the longitudinal axis, the moment of inertia about the x-axis decreases while the moments of inertia about the y and z axes increase. This phenomena increases the coupling between the lateral and longitudinal equations. This can be seen by examining equation 8.2.

$$\frac{\sum M}{I_y} = \ddot{q} + pr \left( \frac{I_x - I_z}{I_y} \right) + (p^2 - r^2) \frac{I_{xz}}{I_y} \quad (8.2)$$

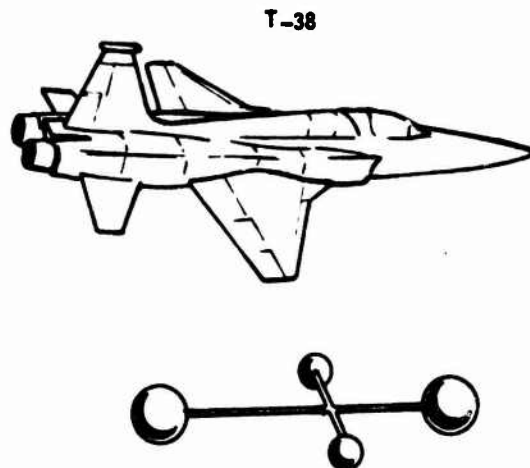
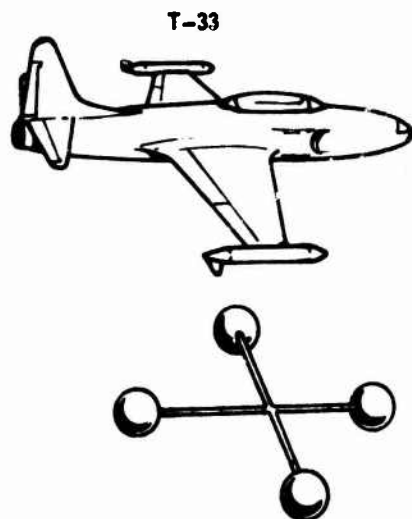
As  $I_x$  becomes much smaller than  $I_z$ , the moment of inertia difference term  $(I_x - I_z)/I_y$  becomes large. If a rolling moment is introduced, the term  $pr (I_x - I_z)/I_y$  may become large enough to cause an uncontrollable pitching moment.

Modern fighter design is characterized by a long slender, high-density fuselage with short, thin wings. This results in a roll inertia which is quite small in comparison to the pitch and yaw inertia. The more conventional low speed airplane may have a wingspan greater than the fuselage length, and a great deal of weight concentrated in the wings. A comparison of these configurations is presented in figure 8.1.

It can be seen that the conventional design presents considerable resistance to rotation about the x-axis. Thus, with this design, one would not expect high roll rates. On the other hand, it can be seen that the modern design presents a relatively small resistance to rotation about the x-axis. Thus, with this design, one could expect to attain high rates of roll. It has been shown that high roll rates

**Figure 8.1**

**CONVENTIONAL AND MODERN AIRCRAFT DESIGNS**

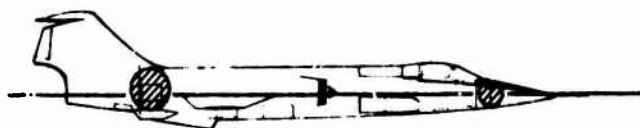


enhance the tendency toward inertial coupling.

This analysis of inertia coupling will consider rolls about two different axes: The inertia axis, and the aerodynamic axis. The inertia axis is formed by a line connecting the aircraft's two "centers of inertia." Refer to figure 8.2.

**Figure 8.2**

**AIRCRAFT INERTIA AXIS**



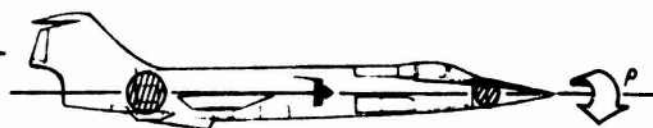
The aerodynamic axis is simply the stability x-axis first introduced in the investigation of the left hand side of the equations of motion. It is merely the line of the relative wind. Aircraft rotation in a roll is generally assumed to be about this axis. To visualize this, recall that to produce a rolling moment, a differential in lift must

be created on the wings. By definition, the differential lift created must be perpendicular to the relative wind. Therefore, the aircraft will roll about the relative wind, or aerodynamic axis.

First, consider a roll when the aerodynamic and inertia axis are coincident. Figure 8.3.

**Figure 8.3**

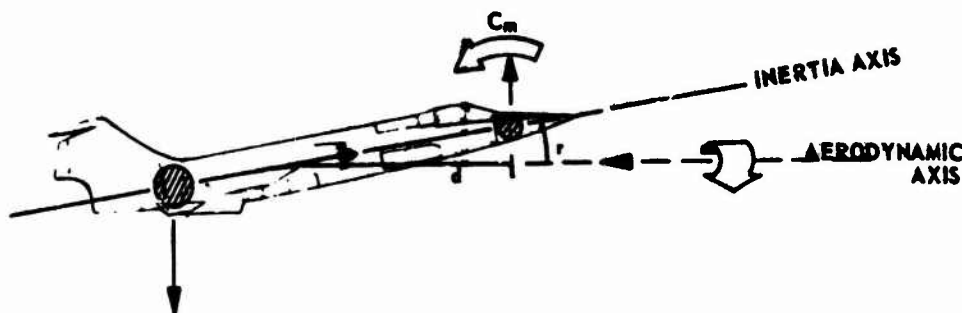
**AERODYNAMIC AND INERTIA AXIS COINCIDENT**



In this case, there is no force created by the centers of inertia that will cause the aircraft to be diverted from its intended flight path, and no inertia coupling results. Now, observe what happens when the inertia axis is displaced from the aerodynamic axis.

Figure 8.4

**AERODYNAMIC AND INERTIA AXIS NON-COINCIDENT**



As the aircraft is rotated about the aerodynamic axis, centrifugal force will act on the centers of inertia. Remembering that centrifugal force acts perpendicular to the axis of rotation, it can be seen that a moment will be created by this centrifugal force. For the case depicted in figure 8.4, where the aerodynamic axis is depressed below the inertia axis, a pitch up will result. Conversely, if the aerodynamic axis is above the inertia axis, a pitch down will result.

To appreciate the magnitude of the moment thus developed, refer to figure 8.4 and consider the following:

$$\text{Centrifugal Force} = \frac{mV_T^2}{r} \quad (8.7)$$

$$V_T = r\omega = rp \quad (8.8)$$

Therefore,

$$\text{C.F.} = mrp^2$$

the moment created by this centrifugal force is

$$M = (\text{C.F.})(d) = mrp^2 d \quad (8.9)$$

For modern designs,  $m$  is large. Also,  $r$  will be larger than for a high aspect ratio wing. (The aircraft will operate at a higher angle of attack.) As previously discussed,  $p$  will be large. Also, for long fuselages,  $d$  will be large. Thus, the moment created by inertia coupling will be large.

**8.3 THE  $I_{xz}$  PARAMETER**

Three products of inertia  $I_{xy}$ ,  $I_{yz}$  and  $I_{xz}$  appear in the equations of motion for a rigid aircraft. By virtue of symmetry,  $I_{xy}$  and  $I_{yz}$  are both equal to zero. However, the product of inertia  $I_{xz}$  can be of an appreciable magnitude and can have a significant effect on the roll characteristics of an aircraft.

The parameter,  $I_{xz}$ , can be thought of as a measure of the uniformity of the mass distribution about the  $x$ -axis. The axis about which  $I_{xz}$  is equal to zero is defined as the principle inertia axis, and the mass of the aircraft can be considered to be concentrated on this axis.

If the  $I_{xz}$  parameter is not equal to zero, then the principle inertia axis is not aligned with the aircraft  $x$ -axis. A typical modern aircraft design can be represented by two centers of mass in the  $xz$  plane designated  $m_1$  and  $m_2$  in figure 8.5.

It can be seen that if the aircraft is rolled about the aerodynamic axis, a pitch down will result. This phenomena is produced by exactly the same centrifugal force effects that produced inertial coupling. However, it should be noted that in this case the x-axis is aligned with the aerodynamic axis and that the pitching moment is a result of the inclination of the principle inertia axis. Thus, when an aircraft is rolled about any axis which differs from its principle inertia axis, pitching moments develop which tend to cause the aircraft to depart from its intended flight path. Depending on its orientation, the  $I_{xz}$  parameter can either aggravate or oppose inertial coupling.

To appreciate the magnitude of this parameter, consider figure 8.6.

Figure 8.5

PRINCIPLE INERTIA AXIS

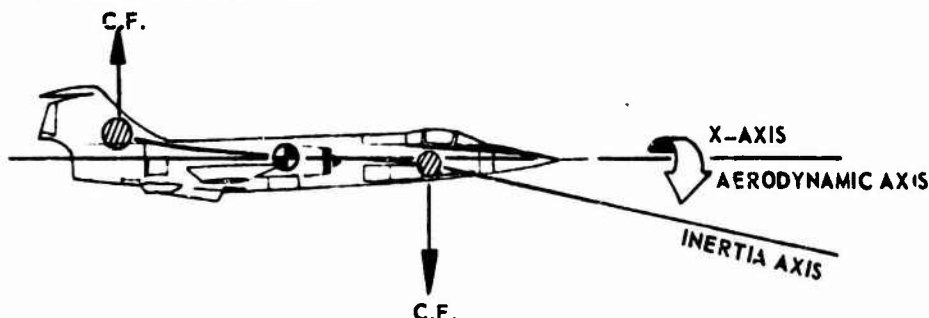
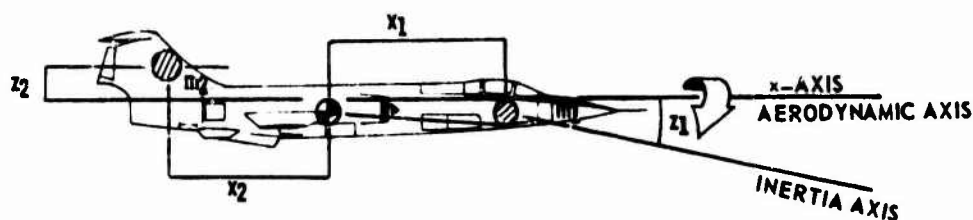


Figure 8.6

PRINCIPLE INERTIA AXIS BELOW AERODYNAMIC AXIS



From equation 8.9, the moment produced by the forward center of mass is,

$$M_1 = (C.F.)(x_1) = m_1 x_1 p^2 z_1 \quad (8.10)$$

Similarly, the moment produced by the aft center of mass is,

$$M_2 = m_2 x_2 p^2 z_2 \quad (8.11)$$

The total pitch moment is therefore,

$$M_T = m_1 + m_2 = m_1 x_1 p^2 z_1 + m_2 x_2 p^2 z_2 \quad (8.12)$$

$$M_T = p^2 (m_1 x_1 z_1 + m_2 x_2 z_2) \quad (8.13)$$

But for a simplified system,

$$I_{xz} = m_1 x_1 z_1 + m_2 x_2 z_2 \quad (8.14)$$

Therefore,

$$M_T = p^2 I_{xz} \quad (8.15)$$

Thus, it can be seen that the magnitude of the pitching moment thus developed depends on the roll rate and the magnitude of the  $I_{xz}$  parameter relative to the roll axis.

The product of inertia,  $I_{xz}$ , is not only of concern because of its introduction of a pitching moment, but also because it plays an important role in determining the aircraft's cross over speed from adverse to complimentary yaw. Primarily, an aircraft's cross over speed is determined by the relationship of the induced and parasite drag generated by a wing during a rolling maneuver. However, this primary effect is mitigated somewhat by the effect of the  $I_{xz}$  parameter.

As previously shown, the  $I_{xz}$  parameter can cause a pitching moment during a rolling maneuver. However, since the aircraft is rolling, gyroscopic effects come into play, and this moment results in a yawing motion. From figure 8.6, it can be seen that when the principle inertia axis is below the roll axis, a pitch down will result. For the right roll depicted, this pitching moment would result in a yaw left. Thus, in this case, the  $I_{xz}$  parameter would contribute to the adverse yaw tendency of the aircraft. In actual practice however, it will be found that the aerodynamic axis generally lies below the principle inertia axis throughout most of the flight envelope. Thus the  $I_{xz}$  parameter will generally cause the aircraft to transition from adverse to complimentary yaw at an earlier speed. For a relative comparison of values, the

F-102 has an  $I_{xz}$  of 3,500 slugs-foot<sup>2</sup> and a crossover speed of 268 knots. The F-100 has an  $I_{xz}$  of 942 slugs-foot<sup>2</sup> and experiences complimentary yaw in roll above 360 knots.

## 8.4 AERODYNAMIC COUPLING

This analysis of roll coupling is not concerned with all aerodynamic coupling terms ( $C_{np}$ ,  $C_{n\delta_a}$ ,  $C_{\ell r}$ ,  $C_{\ell \delta_r}$ , etc.). Only the "kinematic coupling" aspects of aerodynamic coupling will be considered.

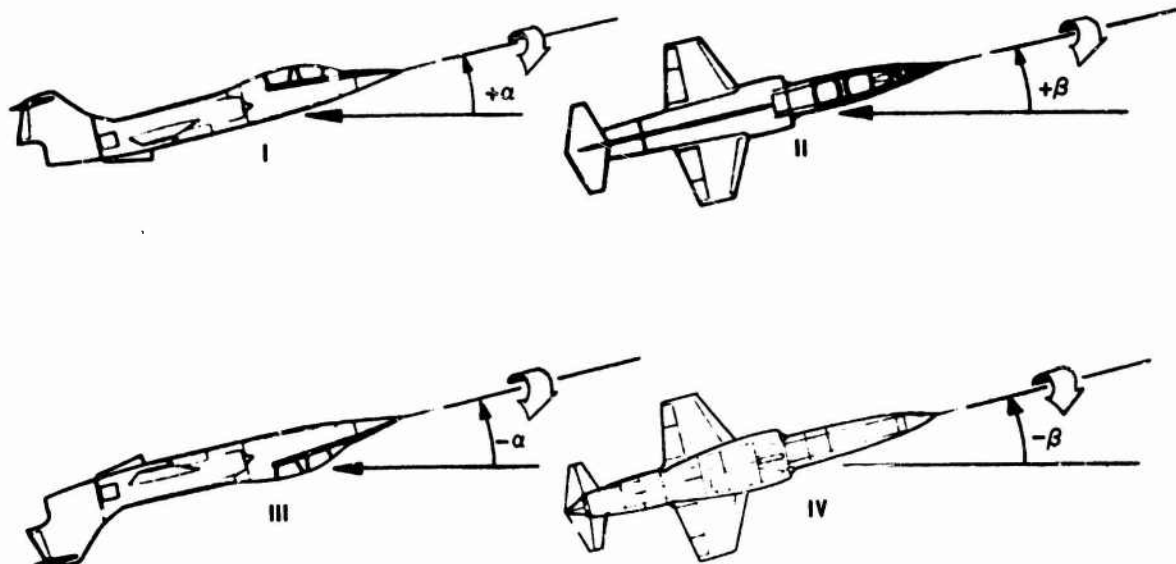
Kinematic coupling may be considered as an actual interchange of  $\alpha$  and  $\beta$  during a rolling maneuver. This interchange is an important means by which the longitudinal and lateral motions are capable of influencing each other during a rapid roll.

To understand how this interchange of  $\alpha$  for  $\beta$  occurs, consider figure 8.7.

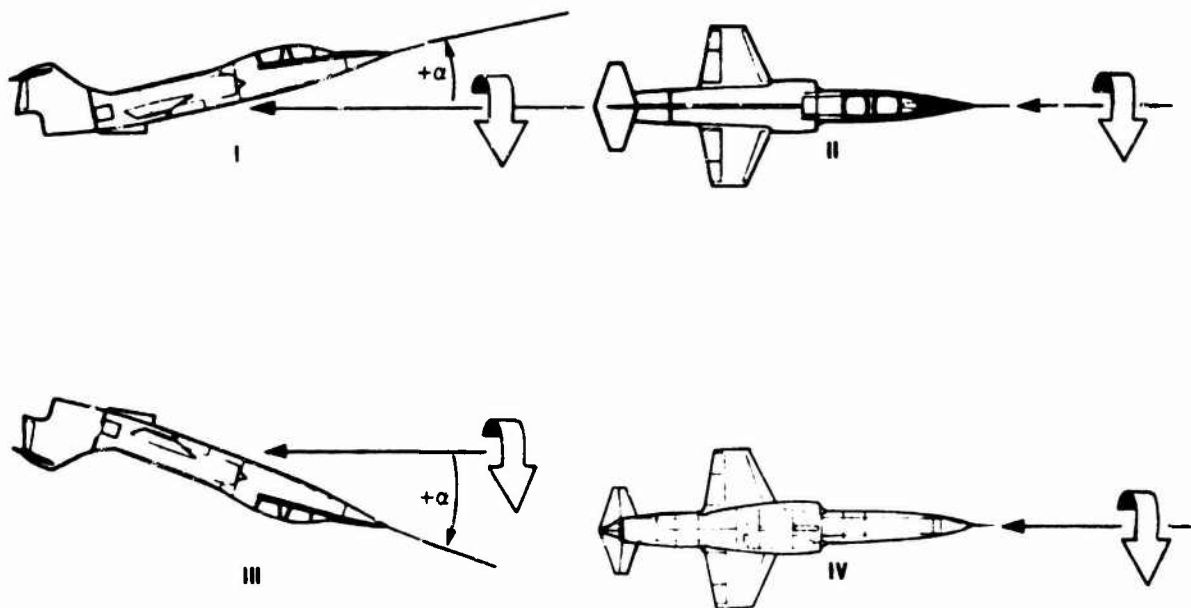
In this figure the aircraft is assumed to have either infinitely large inertia or negligible stability. Thus, it will roll about its principle inertia axis. In (I) the aircraft initiates a roll from a positive angle of attack. In (II) the initial angle of attack is converted to a positive sideslip angle of equal magnitude after 90° of roll. In (III) the aircraft has again exchanged  $\beta$  and  $\alpha$  and after 180° of roll has an angle of attack equal in magnitude but opposite in sign to the original  $\alpha$ . The interchange continues and in (IV) this  $-\alpha$  is converted to  $-\beta$ .

Next, consider an aircraft with infinitely large stability in pitch and yaw or negligible inertia. Refer to figure 8.8.

**Figure 9.7**  
**KINEMATIC COUPLING. STABLE ROLLING OF AN AIRCRAFT WITH**  
**INFINITELY LARGE STABILITY IN PITCH AND YAW OR NEGLIGIBLE INERTIA**



**Figure 9.8**  
**KINEMATIC COUPLING. STABLE ROLLING OF AN AIRCRAFT WITH**  
**INFINITELY LARGE STABILITY IN PITCH AND YAW OR NEGLIGIBLE INERTIA**



In this case, the aircraft will roll about its aerodynamic axis, and no interchange of  $\alpha$  or  $\beta$  will occur. However, in this situation it is possible for inertia coupling to occur.

The situation depicted in figure 8.8 never actually occurs but can be fairly well approximated by an aircraft possessing large magnitudes of  $C_{N\beta}$  and  $C_{m\alpha}$  which is rolled at a relatively slow roll rate. However, in most cases, the actual aircraft motion will lie somewhere between the two extremes depicted.

Two empirical relationships can be stated:

$$\dot{\alpha} = -K_{\beta} \beta \quad (8.16)$$

$$\dot{\beta} = K_{\alpha} \alpha \quad (8.17)$$

These relationships show that any roll rate will cause an interchange of  $\alpha$  and  $\beta$ , the exact amount depending on the relative values of the moments of inertia and  $C_{m\alpha}$  and  $C_{N\beta}$ . It can also be seen that for a given aircraft, the rate of interchange of  $\alpha$  and  $\beta$  depends on the roll rate.

It has been shown that whenever  $\alpha$  and  $\beta$  exist, a rolling maneuver will generate inertia coupling. The relative value of  $C_{m\alpha}$  and  $C_{N\beta}$  will determine just what axis an aircraft will actually roll about, and thus how much interchange of  $\alpha$  and  $\beta$  will occur.

Subsequent to a disturbance in pitch or yaw from an aircraft's equilibrium condition, a finite period of time will be required for the natural aircraft stability to reduce the disturbance to zero. The frequency of this response is a function of the value of  $C_{N\beta}$  and  $C_{m\alpha}$ .

It will be shown in dynamics that,

$$\dot{f}_{N\beta} = (f) C_{N\beta} \quad (8.18)$$

$$\dot{f}_{m\alpha} = (f) C_{m\alpha} \quad (8.19)$$

Assume that an aircraft is rolled at a rate that creates a disturbance in  $\beta$  at a rate equal to the maximum rate that the natural aircraft stability can damp out the disturbance. Thus,

$$\dot{\beta} = K_{\alpha} \alpha = \dot{f}_{N\beta} \quad (8.20)$$

In this case there would be no buildup of  $\beta$ , and a condition of neutral stability in yaw would result. However, if the roll rate were increased slightly above this value then successively larger increases in  $\beta$  would occur and divergence would result. This analysis can also be followed through for an initial disturbance in  $\alpha$ . It is not important which diverges first,  $\alpha$  or  $\beta$ , since any divergence about one axis will quickly drive the other divergent. As a matter of interest however, supersonically  $C_{N\beta}$  decreases more rapidly than  $C_{m\alpha}$  and therefore, most modern aircraft will diverge in yaw first.

It can be shown on an analog computer that when  $C_{m\alpha} = C_{N\beta}$  a stable condition will exist at all roll rates. This is often referred to as a "tuned condition," and is a possible dodge for an aircraft designer to utilize in a critical flight area. However, it is difficult to capitalize on this occurrence because of the wide variation of the stability derivatives with Mach number.

It may be that an aircraft will possess stability parameters such that a roll coupling problem exists at a given roll rate. How-

ever, if a relatively long time is required before large values of  $\alpha$  and  $\beta$  are generated, then the aircraft may be rolled at the maximum value by restricting the aircraft to one 360 degree roll. In this situation, the aircraft is diverging during the roll, but at such a slow rate that by the time the aircraft has rolled 360 degrees, the maximum allowable  $\alpha$  or  $\beta$  of the aircraft has not been exceeded.

### 8.5 AUTOROTATIONAL ROLLING

It has been shown that during rolling maneuvers large angles of sideslip may occur as a result of roll coupling and kinematic coupling. At negative angles of attack, kinematic coupling may cause the vertical tail to produce large rolling moments in the original direction of roll. In such a case, it may not be possible to stop the aircraft from rolling, although the lateral control is held against the roll direction. This is known as autorotational rolling. In this situation, positive "G" would facilitate recovery. As the angle of attack is increased to a positive value, kinematic coupling will result in a moment that opposes the original direction of roll, thus alleviating the tendency for autorotational rolling.

For an appreciation of roll coupling difficulties, computer studies have demonstrated that in quite realistic designs, the critical roll rate for the occurrence of such phenomenon as autorotational rolling can be as low as 20 degrees per second.

### 8.6 A MATHEMATICAL ANALYSIS OF ROLL DIVERGENCE

The roll coupling characteristics of high performance modern fighters are thoroughly investigated by analog simulation prior to

flight testing. However, smaller test programs may not have this capability. It is the intent of this section to provide the test pilot with a practical mathematical tool to aid in determining the roll rate at which an aircraft will start to diverge due to roll coupling. If the critical roll rate thus determined is attainable in the aircraft, then the test pilot should avoid higher rates of roll until a complete analog analysis can be conducted. This, this mathematic tool will enable the test pilot to identify potentially hazardous areas.

This mathematical analysis is based on certain simplifying assumptions. They are:

1. Velocity remains constant during the roll maneuver,  $\dot{u} = 0$ .
2. Roll rate is constant,  $\dot{p} = 0$ .
3.  $v, w, q, r$  are small therefore their products are negligible.
4. Engine gyroscopic effects are negligible.
5. The rudder and elevator are fixed in their initial trim position.
6. Aerodynamic parameters are negligible with the exception of  $M_\alpha, M_q, N_\beta, N_p$ .

When these assumptions are applied to the equations of aircraft motion, the following results are obtained:

$$\frac{\sum F_x}{m} = \dot{p} + q\dot{w} - r\dot{v} = 0 \quad (8.4)$$

$$\frac{\sum L}{I_x} = \dot{p} + q\dot{r} \left( \frac{I_z - I_y}{I_x} \right) - (r + qp) \frac{I_{xz}}{I_x} \quad (8.1)$$

It can be shown that:

$$(\dot{r} + qp) = -\dot{\theta} \sin \theta \approx 0 \quad (8.21)$$

Therefore, in light of the assumptions made, equation 8.1 will be approximately equal to zero.

$$\frac{\sum M}{I_y} = \dot{q} + pr \left( \frac{I_x - I_z}{I_y} \right) + (p^2 - \dot{r}^2) \frac{I_{xz}}{I_y} \quad (8.2)$$

Thus,

$$M_\alpha \cdot \alpha + M_q \cdot q = \dot{q} + pr \left( \frac{I_x - I_z}{I_y} \right) + p^2 \frac{I_{xz}}{I_y} \quad (8.22)$$

$$\frac{\sum N}{I_z} = \dot{r} + pq \left( \frac{I_y - I_x}{I_z} \right) + (q^2 - \dot{p}^2) \frac{I_{xz}}{I_z} \quad (8.3)$$

Thus,

$$N_\beta \cdot \beta + N_r \cdot r = \dot{r} + pq \left( \frac{I_y - I_x}{I_z} \right) \quad (8.23)$$

The lift and side force equations will average zero throughout a roll.

$$\frac{\sum F_z}{m} = \dot{w} + pv - qu = 0 \quad (8.5)$$

$$\frac{\sum F_y}{m} = \dot{v} + ru - pw = 0 \quad (8.6)$$

To get equations 8.5 and 8.6 into a more suitable form, recall that

for small  $\alpha$  and  $\beta$ ,

$$\alpha = \frac{w}{u} \quad (8.24)$$

$$\beta = \frac{v}{u} \quad (8.25)$$

Assuming  $\dot{u} = 0$

$$\dot{\alpha} = \frac{\dot{w}}{u} \quad (8.26)$$

$$\dot{\beta} = \frac{\dot{v}}{u} \quad (8.27)$$

Thus, equation 8.5 becomes,

$$\frac{\dot{w}}{u} + p \frac{v}{u} - q = \dot{\alpha} + p\beta - q = 0 \quad (8.28)$$

Equation 8.6 becomes,

$$\frac{\dot{v}}{u} + r - p \frac{w}{u} = \dot{\beta} + r - p\alpha = 0 \quad (8.29)$$

To further streamline equations 8.22 and 8.23, the following substitutions are made,

$$A = \frac{I_y - I_x}{I_z}$$

$$B = \frac{I_z - I_x}{I_y}$$

To recap, simplifying assumptions have reduced the aircraft equations of motion to the following for rolling maneuvers:

$$\dot{\alpha} + p\beta - q = 0 \quad (8.28)$$

$$\dot{\beta} + r - p\alpha = 0 \quad (8.29)$$

$$M_\alpha \cdot \alpha + M_q \cdot q - \dot{q} + prB = p^2 \frac{I_{xz}}{I_y} \quad (8.30)$$

$$N_{\beta} \cdot \beta + N_r \cdot r - \dot{r} - pqA = 0 \quad (8.31)$$

These equations can be recognized as four linear differential equations expressed in terms of the variables  $\alpha$ ,  $\beta$ ,  $q$ ,  $r$ . To examine the stability of these equations, it is only necessary to find out if the transient solution of the equations delays with time. To do this, first express the equations in Laplace notation:

$$s\alpha + p\beta - q = 0 \quad (8.32)$$

$$s\beta + r - p\alpha = 0 \quad (8.33)$$

$$M_{\alpha} \cdot \alpha + (M_q - s)q + prB = 0 \quad (8.34)$$

$$N_{\beta} \cdot \beta + (N_r - s)r - pqA = 0 \quad (8.35)$$

The characteristic equation of this set of simultaneous, non-homogeneous, differential equations is identical to the determinate of the coefficients. The expanded determinate yields:

$$\begin{aligned} & s^4 \quad (8.36) \\ & + (-M_q - N_r) s^3 \\ & + (-M_{\alpha} + M_q N_r + N_{\beta} + p^2 + p^2 AB) s^2 \\ & + (M_{\alpha} N_r - M_q N_{\beta} - M_q p^2 - N_r p^2) s \\ & + M_{\alpha} p^2 A + M_q N_r p^2 - N_{\beta} B p^2 + AB p^4 - M_{\alpha} N_{\beta} \\ & = 0 \end{aligned}$$

For convenience, the following substitutions will be made:

$$a_3 = -M_q - N_r \quad (8.37)$$

$$a_2 = -M_{\alpha} + M_q N_r + N_{\beta} + p^2 (1 + AB) \quad (8.38)$$

$$a_1 = M_{\alpha} N_r - M_q N_{\beta} - p^2 (M_q + N_r) \quad (8.39)$$

$$a_0 = p^2 (M_{\alpha} A + M_q N_r - N_{\beta} B + AB p^2) - M_{\alpha} N_{\beta} \quad (8.40)$$

Thus, equation 8.36 becomes

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \quad (8.41)$$

If any roots of equation 8.41 have positive real parts, then the motion will be unstable and the aircraft will diverge. In order that there be no roots with positive real parts, it is necessary but not sufficient that:

1. All coefficients have the same sign.
2. None of the coefficients vanish.

If both of these conditions are met, then the equation must be examined further.

The coefficients of the characteristic equation will be examined in light of the following assumptions:

1. The aircraft possesses positive static stability in pitch and yaw, thus  $M_{\alpha} = (-)$ ,  $N_{\beta} = (+)$ .
2. The aircraft possesses positive damping in pitch and yaw, thus,  $M_q = (-)$ ,  $N_r = (-)$ .
3. The aircraft mass distribution is such that  $I_z > I_x$  and  $I_y > I_x$ , thus  $A = (+)$ ,  $B = (+)$ . (This mass distribution is typical of a modern fuselage loaded aircraft.)

In view of these assumptions, it can be seen that the coefficients  $a_2, a_2, a_1$  will be positive. However,  $a_0$  may be either positive or negative. If  $a_0$  is examined and found to be negative, the resulting rolling maneuver will be unstable. If, however,  $a_0$  is found to be positive, the coefficients must be investigated further by means of the Routh-Hurwitz Stability Criterion. A brief description of the mech-

anics involved in the Routh-Hurwitz method follows:

The first step is to arrange the polynomial coefficients into two rows: The first row will consist of the first, third, fifth coefficients, etc., and the second row will consist of the second, fourth, sixth coefficients, etc. The following example is presented:

$$\begin{array}{ccccccccc} a_0 & & a_2 & & a_4 & & a_6 & & a_8 \\ a_1 & & a_3 & & a_5 & & a_7 & & a_9 \end{array}$$

The next step is to form the following array of numbers obtained by the indicated operations. The example shown is for a sixth-order system.

$$F(s) = a_0 s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s^1 + a_6$$

$$\begin{array}{ccccccc} a_0 & & a_2 & & a_4 & & a_6 \\ a_1 & & a_3 & & a_5 & & 0 \\ \frac{a_1 a_2 - a_3 a_0}{a_1} = A & & \frac{a_1 a_4 - a_5 a_0}{a_1} = B & & \frac{a_1 a_6 - a_0 \times 0}{a_1} = a_6 & & 0 \\ \frac{A a_3 - a_1 B}{A} = C & & \frac{A a_5 - a_1 a_6}{A} = D & & \frac{A \times 0 - a_1 \times 0}{A} = 0 & & 0 \\ \frac{CB - AD}{C} = E & & \frac{C a_6 - A \times 0}{C} = a_6 & & \frac{C \times 0 - A \times 0}{C} = 0 & & 0 \\ \frac{ED - C a_6}{E} = F & & 0 & & 0 & & 0 \\ \frac{F a_6 - E \times 0}{F} = a_6 & & 0 & & 0 & & 0 \end{array}$$

The last step is to investigate the signs of the numbers in the first column in this array. If all of the elements in the first column are positive, the system is stable. If one or more of the elements is negative, the system is unstable.

When the Routh-Hurwitz Criterion is applied to equation 8.41, the following equations result. They must both be positive if the system is to be stable.

$$a_3 a_2 - a_1 = 0 \quad (8.42)$$

$$a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 = 0 \quad (8.43)$$

In summary, to determine if a given roll rate will result in a stable aircraft motion:

1. Determine the value of  $a_0$  (equation 8.40). If it is negative the system is unstable.
2. Determine the values of  $a_1$ ,  $a_2$ ,  $a_3$  (equations 37 - 39).
3. Solve equation 8.42. If negative the system is unstable.
4. Solve equation 8.43. If negative the system is unstable.
5. If steps 1-4 yield no negative results, the system is stable for the roll rate investigated.

Although somewhat unwieldy, in the absence of adequate analog simulation, the foregoing system will yield fairly accurate results. It can serve to warn the test pilot of a perilous situation.

## 8.7 CONCLUSIONS

As an aircraft's inertias are disproportionately increased in relation to its aerodynamic stabilities in pitch and yaw, the aircraft will be liable to pitching and yawing motions during rolling maneuvers. The more typical case is a divergence in yaw by virtue of an inadequate value of  $n_{\beta}$ .

The peak loads resulting from roll coupling generally increase in proportion to the initial incidence of the principle inertial axis and progressively with the duration of the roll and the rapidity of aileron application at the beginning and the end of the maneuver. The most severe cases naturally should be expected in a flight regime of low  $C_{n\beta}$  and high dynamic pressures.

The rolling pull-out maneuver in a high performance aircraft is especially dangerous. It combines many unfavorable features: High speed hence high roll rate capability; high acceleration which favors poor coordination and inadvertent excitation of transients by the pilot; and high dynamic pressures which at large values of  $\alpha$  and  $\beta$  may break the aircraft.

Most high performance aircraft incorporate roll rate limiters in addition to angular damping augmenters. In these aircraft a lateral control with enough power for low speed is almost certain to be too powerful for high speeds. Fortunately, limiters of various kinds are not too difficult to incorporate in a fully powered control system.

It is obvious that flight testing in suspected regions of roll coupling warrant a cautious methodical approach and must be accompanied by thorough computer studies that stay current with the flight test data. The only way that the pilot can discover the

exact critical roll limit in flight is when he exceeds it, which is obviously not the approach to take. Because of this, flight tests are generally discontinued when computer studies indicate that the next data point may be "over the line."

The following example is cited. The Bell X-2 rocket ship in 1956 was launched from its mother ship at Edwards. The pilot flew a perfect profile but the rocket engine burned a few critical seconds longer than the engineers predicted, resulting in a greater speed (Mach 3.2) and greater altitude (119,300 feet) than planned. Unknown to the pilot, he was progressively running out of directional stability. When he was over the point at which he had preplanned to start his turn toward Roger's Dry Lake he actuated his controls. The X-2 went divergent with a resultant loss of control. The accident investigation revealed the cause to be a greater loss in directional stability than planned, resulting in divergent roll coupling.

A combination of reasonable piloting restrictions coupled with increased directional stability has provided the solution to roll coupling problems in the present generation of aircraft. The problem is one of understanding since a thinking pilot would no more exceed the roll limitations imposed on an aircraft than he would the structural "G" limitations.

Besides pilot education, some other schemes to eliminate roll coupling divergencies are:

1. Roll rate limiters.
2. Angular damping augmenters.
3. Placarded roll limits such as
  - a. "G" limits.
  - b. Total allowable roll at maximum rate.
  - c. Altitude limits.
  - d. Mach limits.
  - e. Flap position limits.

**CONTROL SYSTEMS**

**ABBREVIATIONS AND SYMBOLS  
FOR THIS CHAPTER**

$C_{h_\delta}$	hinge moment coefficient (restoring)	
$C_{h_\alpha}$	hinge moment coefficient (floating)	
$C_{h_{\delta_t}}$	hinge moment coefficient (tab)	
$q$	"dynamic pressure"	lbs/ft <sup>2</sup>
$q_c$	compressible dynamic pressure	lbs/ft <sup>2</sup>
$\delta$	elevator deflection	degrees or radians
$V$	airspeed	knots
$d\delta/dn$	elevator angle per g	
$n$	normal acceleration	ft/sec <sup>2</sup>
$dF_s/dn$	stick force per g	
$d\delta_e/dv$	elevator angle to airspeed ratio	
$d\delta_r/d\beta$	rudder angle per degree of sideslip	
$\delta_e$	elevator deflection	degrees or radians
$C_{n_r}$	Yaw damping derivative	
$C_{n_\beta}$	Yawing moment coefficient with sideslip angle	
$C_{n_p}$	Yawing moment coefficient with rolling velocity	

## 9.1 INTRODUCTION

In the broad sense the aircraft flight control system consists of all the mechanical, electrical and hydraulic elements which convert cockpit control forces and motions into aerodynamic control surface deflections or action of other control devices which in turn change the orientation of the vehicle. The flight control system together with the power plant control system, enables the pilot to "fly" his aircraft - that is, to place it at any desired flight condition within its capability.

The power plant control system acts as a thrust metering device, while the flight control system varies the moments about the aircraft center of gravity. Through these control systems the pilot is able to vary the velocity, normal acceleration, sideslip, roll rate, and other parameters within the aircraft's envelope. How easily and effectively he can accomplish his task is a measure of the suitability of his control systems. An aircraft with exceptional performance characteristics is virtually worthless if it is not equipped with at least an acceptable flight control system.

Two important control system characteristics are the magnitude of cockpit control forces and deflections. Figure 9.1 shows approximate limitations of the pilot's physical effort. The limitations shown in this figure are much greater than those considered desirable for normal flying. However, if the control forces required for normal aircraft maneuvers are pleasantly light it will usually be possible to overstress the aircraft by misuse of the controls. Conversely, an aircraft that displays control characteristics that prevent exceeding the design limitations would ordinarily be considered unacceptably heavy.

FIGURE 9.1  
APPROXIMATE PILOT PHYSICAL LIMITATIONS

CASE	AILERON STICK WHEEL		ELEVATOR STICK WHEEL		RUDDER
MAX ALL OUT EFFORT FOR VERY SHORT TIME (2 HANDS)	90	120	100	120	400
MAX PERMISSIBLE (2 HANDS) FOR SHORT TIME (1 HAND)	50	80	100	110	300
MAX COMFORTABLE (1 HANDS) FOR SHORT TIME (1 HAND)	30	50	70	80	80
LARGEST HAND OR FOOT MOVEMENT FOR FULL TRAVEL	-10 IN -20 IN		-9 IN -9 IN		-5 IN

Having noted the approximate capabilities of a pilot with respect to effort, there is another important factor to which he reacts, that of control "harmony." Control harmony is a rather nebulous quantity and cannot be discussed merely in terms of relative forces required for the three controls, it is also necessary to take into account the aircraft's response characteristics. Therefore, no idealized ratios can be established for relative heaviness of the controls. However, as one would suspect after reference to figure 9.1, aileron force should ordinarily be less than elevator force which in turn is usually expected to be less than the rudder force required. In any case control harmony is a matter of opinion, and opinions change with time and pilots. Since the pilot-control system acts on an aircraft with specific static and dynamic stability properties, it follows that the characteristics of the closed loop system must be found satisfactory.

In flying an aircraft the pilot frequently makes use of such references as the movement about the horizon and the magnitude of accelerations felt, but these indications are available to him after he is already in the maneuver. He can only conclude that this maneuver is too violent or too mild after he has felt or seen its magnitude. Needless to say this could be rather untimely for effective control of the aircraft, consequently there is a need for something to forewarn the pilot of aircraft motion that

will take place as a result of his use of the cockpit controls. Control movement and force are two of the pilot's inputs to the stick, of which he is usually aware. In some flight regimes the stick movement can be considerable but at high speeds and aft cg conditions, the stick movements may be barely perceptible and for this reason it is generally conceded that control force is the most important indication of the magnitude of the maneuver. It follows that there should be no reversal of these forces in order that the pilot is never required to reverse his thinking. It would seem that stick deflections should never reverse either, but it may be possible to relax such a requirement in the portion of the aircraft envelope where stick movement is so small as to be unnoticeable. If the stick movement is infinitesimal the pilot simply senses that he must pull to slow down and is probably unaware of the fact that the stick may have moved forward a fraction of an inch.

To summarize the general requirements of the aerodynamic control system; two conditions must be met if the pilot is to be given suitable command over his airplane.

1. It must be capable of actuating the control surface.
2. It must provide the pilot with a "feel" that bears a satisfactory relationship to the aircraft's reaction.

There are numerous variations in the designs of aircraft control systems. However, these systems may be rather simply classified. The first classification in which the majority of the modern aircraft control systems fall is that in which the moments about the aircraft center of gravity are created by varying the camber or angle of attack of the aerodynamic lifting

or stabilizing surfaces. The most familiar example, is the use of trailing edge flaps on the wings, horizontal and vertical stabilizer, called ailerons, elevator and rudder, respectively. Another subclass of aerodynamic control is the spoiler which has come into rather wide use in recent years for lateral control. This device creates rolling moments by changing the energy of the airflow over one wing thereby resulting in a differential lift between the two wings. These aerodynamic systems can be further broken down into "reversible" and irreversible systems. These systems can be simple mechanical controls in which the pilot supplies all of the force required to move the control surface. This type system is called "reversible" since all of the forces required to overcome the hinge moments at the control surface are transmitted to the cockpit controls. The system may have built into it a mechanical, hydraulic or some other type of boosting device, which supplies some specific proportion of the control force. Systems of this nature are generally called "boosted control systems." However, they are still considered "reversible." Even though the force required of the pilot is less than the control surface hinge moments the force required is proportional to these moments. In other words the pilot furnishes a fraction of the force required to overcome the hinge moments throughout the aircraft's envelope. The control system is said to be irreversible if the pilot through his cockpit controls, actuates a hydraulic, electronic, or some other type of device which in turn moves the control surface. In this system the aerodynamic hinge moments at the control surface are no longer transmitted to the stick. Here, without artificial feel devices the pilot would feel only the force required to actuate the valves or sensing devices of his powered control system. Because of this, artificial feel, which approximates the feel that the pilot

senses with a "reversible" system must be added.

A second category of control system is that of reaction controls. In this system small jets are usually located near the extremities of the aircraft where the length of the moment arm is greatest. These jets may be fired in order to create pitching, rolling, or yawing moments. Reaction controls are used extensively for control of vehicles in the very low speed or very high altitude flight regimes where aerodynamic controls are ineffective. Since this control system is used primarily in areas where aerodynamic damping is very light or in some cases almost non-existent, it becomes apparent that a different technique is required in the use of the cockpit controls. When an angular velocity about the cg is established by use of one of the reaction jets, this velocity tends to continue undiminished until an equal and opposite pulse is introduced. Therefore, it is obvious that a considerable change in ground rules is required for the pilot to intelligently evaluate such a control system.

Other control system categories might include the use of inertial or magnetic control devices not only to guide, but to control the orientation of a vehicle in space.

Since in the final analysis it is the pilot-control system - aircraft combination which must be satisfactory in order for the aircraft itself to be considered acceptable, this chapter will concern itself with not only the various types of control systems mentioned above, but also with the pilot himself.

## **9.2 SERVOMECHANISMS**

From experience it has been shown that in simple tasks the difference in performance between

one pilot and another is generally small, becoming negligible when such tasks are repetitive. As the complexity of the task is gradually increased his performance becomes more inconsistent. Then at each progressively more difficult stage the performance depends on the variance of adaptability of pilots, the degrees of predictability of the system they operate, their individual nervous and muscular characteristics and also their psychological and physiological condition at the time. In spite of all this the human pilot is popular with servo engineers for use as an adaptive servo. The human, however, has some limiting characteristics which are virtually constant. They are his reaction time (normally between 0.2 and 0.3 seconds) and his neuro-muscular lag (0.10 to 0.16 seconds). These virtually fixed characteristics represent some of his most serious liabilities as a servo element and restrict system performance. Therefore, the control system designer must design for the physical limitations of the pilot as well as for the airplane characteristics.

A servomechanism can be defined as an automatic control system which senses the output of a system, compares this output with the desired output, and if a difference exists - causes the output to be changed until it equals the desired value. Familiar examples of servomechanisms are automatic tracking radar, and thermostatically controlled air conditioning units. Another example is the human body when performing even the simplest of tasks. A boy catching a ball senses the direction of motion of the ball with his eyes and moves his body, arms and hands until the ball enters his glove. He is subconsciously but continually comparing the relative position and motion of the ball and glove and correcting the motion of the glove until the ball enters it. A pilot flying an aircraft, may be thought of as part

of a multi-channel servomechanism. He is constantly monitoring and - through the flight control system - adjusting angles of pitch and bank, airspeed, altitude, heading, and often a number of other parameters as well.

Before further considering the pilot as a servomechanism, a brief discussion of the servomechanism is in order. The servomechanism is based on the detection of a difference between the existing output of a system and the desired output, and a resulting correction of the output toward the desired value. A signal (such as a voltage), representing the output is "fed back" to a sensing device which detects this difference which is called the error signal. The route by which the existing output is fed back to the sensing device is called the feedback loop. A control system containing one or more feedback loops is called a closed loop system. A system which does not contain a feedback element is called an open loop system. Examples of a closed loop and open loop system are shown in figures 9.2 and 9.3.

FIGURE 9.2  
CLOSED LOOP SYSTEM

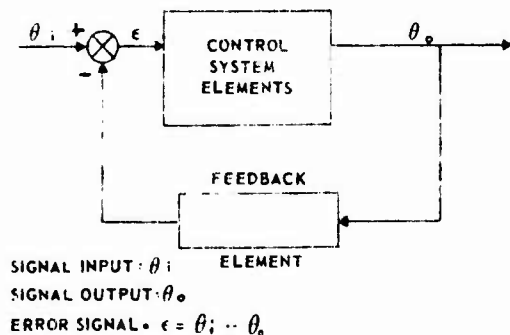


FIGURE 9.3  
OPEN LOOP SYSTEM



An open loop system does not qualify as a servomechanism. An example of an open loop system is an aircraft with the controls free to float. An example of a closed loop system is an aircraft under the control of a pilot. Typical block diagrams for these two systems are shown in figures 9.4 and 9.5.

FIGURE 9.4  
CLOSED LOOP SYSTEM

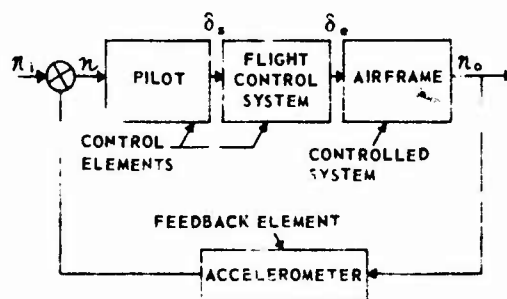
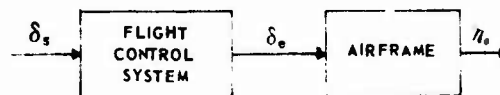


FIGURE 9.5  
OPEN LOOP SYSTEM



In the closed loop case, the input is a desired normal acceleration of say 2.0 g's. If the aircraft is at only 1.8 g's (the output) the pilot reads and interprets the accelerometer and pulls back on the stick. Thus, he is both a sensing and a control element. He senses an error signal of 0.2 g and imparts an aft deflection to the stick. This aft motion of the stick produces a trailing edge up elevator deflection which in turn, produces an increase in normal acceleration. When the actual normal acceleration (the output) equals the desired normal acceleration (the input) the error signal will then be zero and the pilot will stop the aft movement of the stick.

In the open loop system, the stick is deflected to a position which is estimated to produce a normal acceleration of 2.0 g's. The elevator is moved to a corresponding angle by the flight control system, and the airplane is given a normal acceleration which may or may not be exactly 2.0 g's. Since there is no feedback, no further refinement is possible. The normal acceleration obtained with this stick deflection depends on many variables including altitude, airspeed, cg position, etc.

It is quite possible for an aircraft to exhibit satisfactory open loop characteristics and yet be unsatisfactory when the pilot is introduced into the system and the loop is closed. For example, suppose that the stick-fixed longitudinal dynamic characteristics of an aircraft were such that an elevator pulse produced a pitch rate trace as shown in figure 9.6.

FIGURE 9.6  
OPEN LOOP DYNAMIC TIME HISTORY

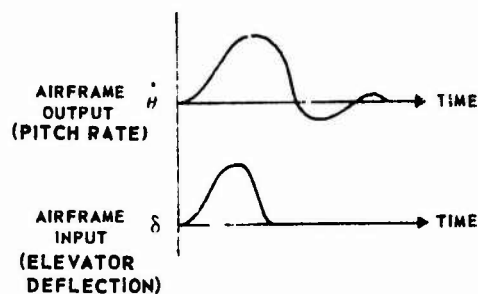
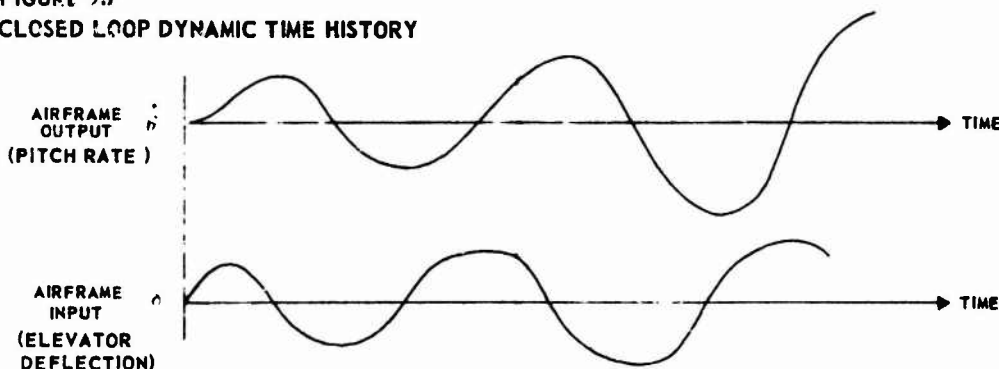


FIGURE 9.7  
CLOSED LOOP DYNAMIC TIME HISTORY



This open loop dynamic time history indicates stable, moderately-damped longitudinal dynamics. If the pilot were required to control the aircraft quite closely, however, such as in low-level, high-speed formation flying, the closed loop dynamic time history might indicate an entirely different situation. The resulting pilot-induced oscillations might lead to normal accelerations which would produce structural damage to the aircraft. The results seen in figure 9.7 could well be caused by the designer failing to take into account that the normal pilot response time was similar to the short period in a portion of the aircraft's envelope.

Thus the job of matching airframe and pilot characteristics with suitable flight control system characteristics is not an easy one. But if the airplane designer does not bear in mind the basic characteristics and limitations of the pilot when he designs the aircraft and control system, the results might well prove disappointing.

### 9.3 BOOSTED CONTROL SYSTEMS

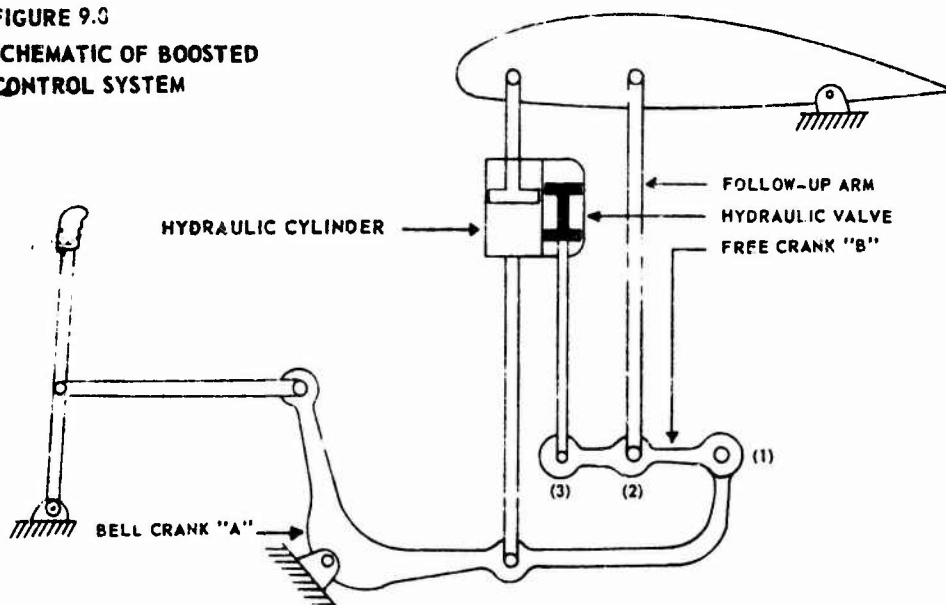
As the size and speed of aircraft increase, the hinge moments and thus the control forces increase accordingly. Neglecting Mach effects, if the dimensions and speed of a given aircraft are both doubled,

the control forces will be multiplied by a factor of thirty-two. Although aerodynamic balancing could alleviate the problem up to a point, the difficulty in the past of achieving the production tolerances necessary to produce a  $C_{H\delta}$  of  $\pm 0.0002$  per degree, and the erratic variations in hinge moment in the transonic regime led to the development of boosted control systems. In this type of system the pilot is required to produce only a fraction of the force required to overcome the hinge moments. However, he does produce a specified portion of this force and the force required of the pilot is still approximately proportional to the hinge moments on the control surface. The control system then, is still a reversible one in that the hinge moments are fed back to the stick.

One method of control boosting is shown in figures 9.8 and 9.9. This system uses the hydraulic method of amplifying which requires the addition of a hydraulic cylinder on a follower arm. As in the unboosted system the pilot moves the stick aft and the trailing edge of the elevator deflects up.

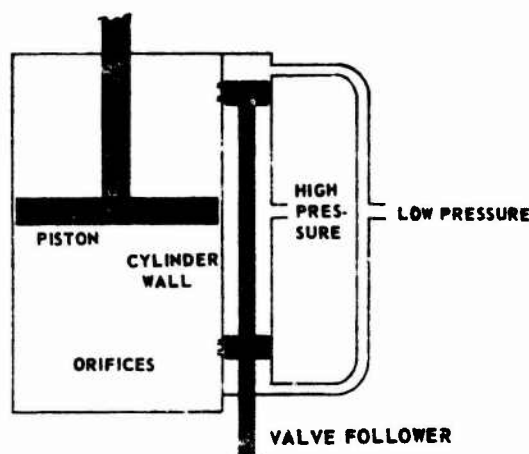
The process however causes a hydraulic pressure drop across the piston causing the cylinder to collapse and affecting additional elevator deflection. As the cylinder collapses a follow-up linkage reduces the pressure drop across the piston to zero and the follower arm contraction ceases. Starting from an initially trimmed condition with zero stick force the pilot pulls back on the stick and holds. There is an accompanying up elevator produced, identical to that which would be produced in an unboosted system. This up elevator moves the follow-up arm down but point (2) does not move as far as point (1) and thus the free crank "B" rotates clockwise relative to bell crank "A". This produces an upward motion at point (3) relative to the horizontal arm of the bell crank "A" and the valve slides toward the elevator, thereby unporting two orifices in the cylinder wall astraddle the piston head. High-pressure oil is fed to the orifice on the rod side and the oil pressure drop across the piston forces it to collapse and therefore shorten the follower arm achieving an additional up elevator deflection. As the elevator moves up, the follow-

FIGURE 9.9  
SCHEMATIC OF BOOSTED  
CONTROL SYSTEM



up arm goes down and point (2) moves down relative to point (1). This action rotates the free crank "B" counterclockwise and the valve slides down on the cylinder until the orifices are closed and the action is complete.

**FIGURE 9.9**  
**HYDRAULIC VALVE**



The hydraulic valve consists of a double piston and cylinder arranged so that a slight motion in either direction of the valve follower opens two orifices straddling the hydraulic cylinder piston head. If the valve follower is forced upward the high-pressure orifice on the rod side and the low-pressure orifice on the face side are opened. If the valve follower is forced downward the reverse action takes place. (See figure 9.9.)

Some advantages of the boosted control system are:

1. Aerodynamic feel from the control surface is retained.
2. In event of power failure the controls can be operated manually by the pilot, at least over a limited flight regime.

Some disadvantages are:

1. Unless the boost ratio is variable, control forces may still be excessive at high speeds (low boost ratio) or too low at low indicated airspeeds (high boost ratio).
2. Normal antifiutter mass balance is still required.
3. A certain amount of aerodynamic balance is still required for manual operation.
4. Power failure in out-of-trim flight will produce very high control forces.

On aircraft which have a wide speed range, the difficulty of avoiding the problems mentioned in (1) makes full power operation of controls desirable.

#### **9.4 POWERED CONTROL SYSTEMS**

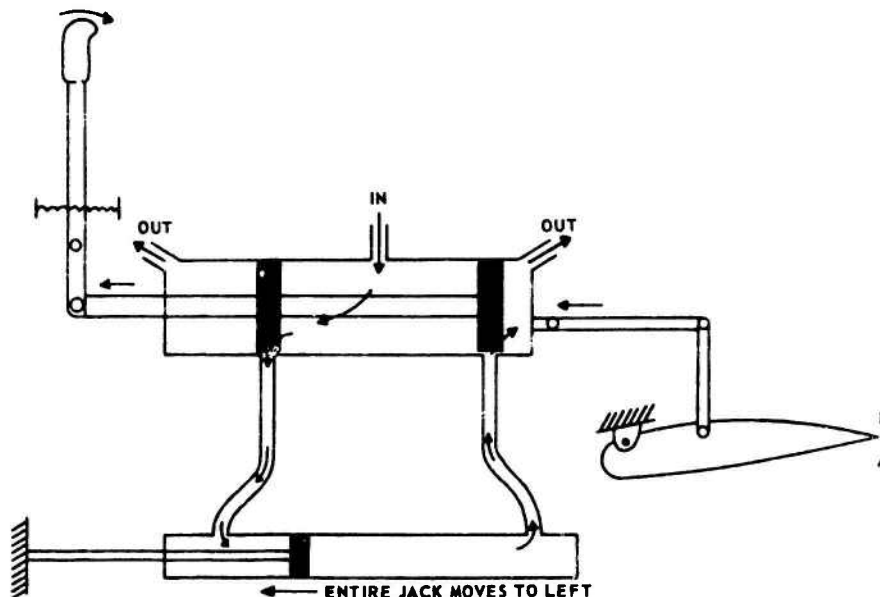
The aircraft designer can, by proper use of aerodynamic balance and/or boost, enable the pilot to cope with control hinge moments of extremely large magnitude. These methods are practical as long as the control derivatives ( $Ch_\alpha$ ,  $Ch_\delta$ ,  $Ch_{\delta_t}$ ) remain essentially constant. But as aircraft approach sonic speed they run into a region where the hinge moments vary widely with very small speed changes, and as they accelerate to supersonic speeds the hinge moment characteristics settle down but at substantially different values than those experienced in subsonic flight. This is due primarily to the aft shift in center of pressure. Therefore, the designer divorces the pilot from the control surface by giving him a control system in which he activates the flight control surface indirectly through a mechanical or electrically actuated system - usually hydraulically operated. A system such as this would require

the pilot to overcome forces which originate from friction of the control linkages, valves, etc., and not directly from control surface movement. It is fairly obvious that these forces would not be a satisfactory measure of aircraft response. Thus the designer must supply the pilot with an artificial feel system. One of the purposes of this section is to discuss some of the relatively simple devices used in irreversible control systems. The following section will deal with the more complex problem of feel systems.

A schematic of a typical powered system is shown in figure 9.10. The pilot, by pulling the stick aft, positions the upper piston or valve in the jack. The incoming hydraulic fluid forces

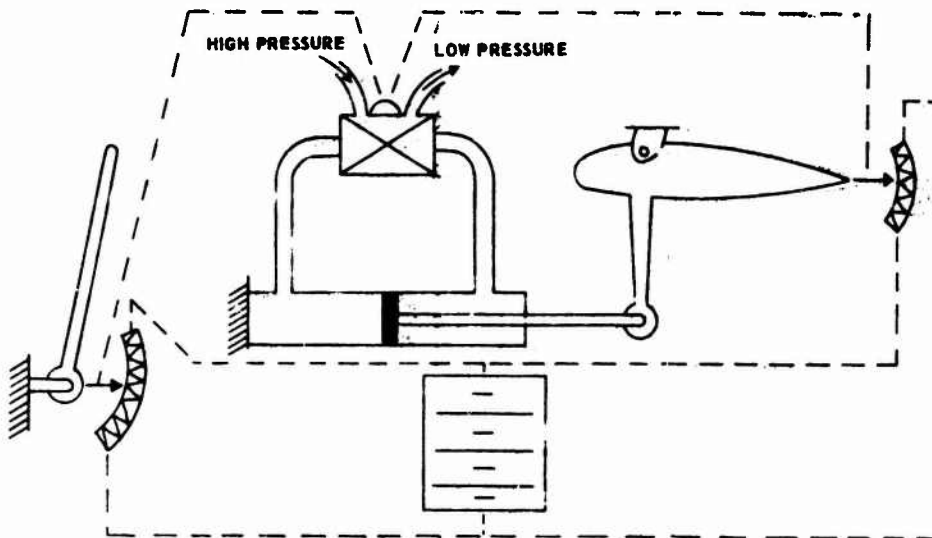
the lower piston to the right with respect to the jack. Since the piston is fixed to the structure, the jack must move to the left. This movement, in turn leaves the control surface trailing edge up. As the jack moves to the left the ports of the upper cylinder are again covered and the control deflection ceases. If the pistons in the upper valve have only a small overlap the control surface will follow the stick deflection quite closely. However, this type of system, while capable of handling very large hinge moments, is not capable of handling unlimited loads. If the aerodynamic hinge moment on the control surface exceeds the mechanical hinge moments which can be generated by the powered control system, the system encounters a "jack stall."

**FIGURE 9.10**  
**POWERED IRREVERSIBLE CONTROL SYSTEM**



A variation of the powered control can be made because it is not necessary to make use of mechanical valve linkage. An electrically operated servo valve can be used to meter the hydraulic fluid to the cylinder, by supplying it with a voltage proportional to stick deflection (see figure 9.11).

**FIGURE 9.11**  
**POWER OPERATED CONTROL WITH ELECTRIC SERVO**



To fulfill the positional requirement of stick-elevator gearing, a feedback voltage in the ratio of elevator deflection is also fed back to the servo. Stick motion is not the only way to obtain control surface deflection. Since only a voltage is required, a stick-mounted strain gage generating voltage proportional to stick force will command surface deflection. It is generally accepted that the power systems using mechanical linkage rather than electrical signals are somewhat more reliable but are less adaptive to sophisticated damping systems and automatic weapons delivery systems. In most cases it is wise to choose the least complex system which is capable of doing the task required.

The most important advantages of using powered irreversible control systems are:

1. The pilot is divorced from the large and often erratic control forces required at high speeds.
2. Aerodynamic balancing is not usually required. The rigidity associated with powered control systems reduces the requirement for mass balancing, although some attention must still be given to the problem in order to avoid high speed flutter.

The primary disadvantages are:

1. An artificial feel system must be incorporated into the aircraft control system.
2. The increased complexity of the control-feel system combination reduces system reliability.
3. It is often difficult or impractical to build into the system a standby capable of reverting to manual reversible control.

## ● 9.5 AIRCRAFT FEEL SYSTEMS

Aircraft feel was discussed at some length in section 9.1, however, several additional comments will now be made before some of the artificial devices that affect feel are described.

The control system characteristics, together with the airframe stability characteristics, are determined by airframe design, the control-feel system is the place where the handling qualities may be patched up - or ruined.

The aircraft-feel system ordinarily includes stability augmentation devices, since they definitely affect aircraft handling qualities, but because of their dual function and unique character they will be considered separately.

The feel characteristics of a reversible control system may certainly be altered in order to improve handling qualities. In fact two devices for doing just that were discussed previously in this course, i.e., the downspring and bobweight. In the following paragraphs, though, the control system will be assumed to be irreversible and feel systems artificial.

## ● 9.6 MECHANICAL CHARACTERISTICS OF CONTROL SYSTEMS

Mechanical characteristics of control systems which affect feel will now be discussed before going into artificial devices which are incorporated in order to improve feel.

### Breakout Force:

This term defines the force necessary to be applied to the stick before it moves. The source lies in the cumulative affect of the following:

1. The mechanical friction of the control circuit, the feel unit, and the valve.
2. The force due to viscous flow past the valve in its neutral position and/or valve centering spring, and
3. The preload of the feel unit.

If the breakout force is high, it will tend to produce overshooting in the desired small and rapid control pressures (e.g., during tracking or instrument flight) because of the pilot's neuromuscular and reaction lag. As the pilot exerts pressure on the cockpit control to overcome this force he is likely to continue this pressure for a short period after the friction is overcome and thus take the control past the desired value. These effects will be aggravated if the force level immediately following breakout force becomes noticeably lower because of lower running friction.

### Backlash:

Mechanical play in the control system resulting from cable stretch, valve overlap, etc., is called by various names - backlash, lost motion, free play, mechanical hysteresis. If it reaches annoy-

ing proportions it is frequently called "slop." The cockpit-control/control-surface relationship is not constant. This phenomenon makes the pilot work even harder to maintain steady flight or make small corrections. The combined effects of breakout force and backlash on a system where the pilot is attempting to make a small correction is shown in figure 9.12.

The pilot starts applying a force at time "0." At time "1" he has applied enough force to overcome the breakout force and the stick starts to move. At time "2" the stick has moved far enough to take up the mechanical play in the system and the control surface starts to move. At time "3" the control surface reaches the desired deflection. Just after this time, the pilot starts to release force, but the control surface has moved past the desired point until time "4"

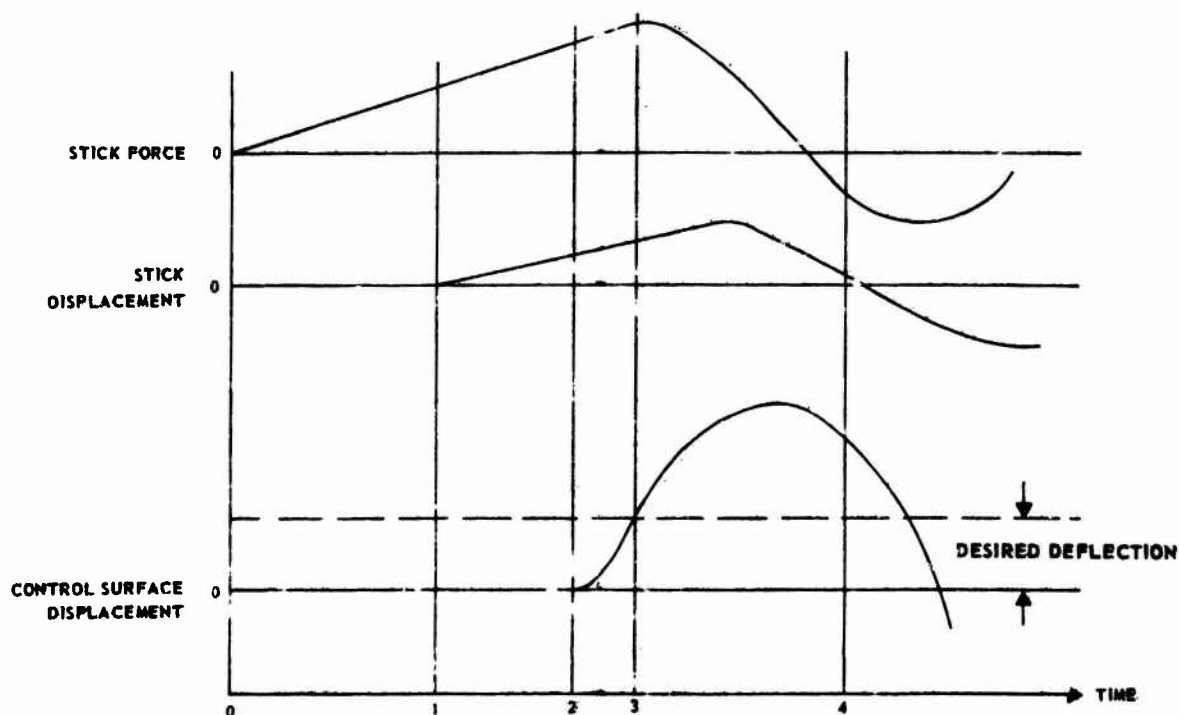
when he has applied enough force in the opposite direction to again overcome friction, backlash, etc. Such a system would thus cause the pilot to overcontrol.

If the initial aircraft response is slow and damping is high, the overshoot is likely to be negligible; however, rapid aircraft response to control pressures will result in marked overshoot, often culminating in pilot induced oscillations. In more severe cases of breaking forces and backlash, exciting landing flareouts may result.

#### Centering:

Good centering of the cockpit control and the control surface is a requirement for any satisfactory aerodynamic control system. Centering may be defined as the degree to which the pilot's control and

**FIGURE 9.12**  
**EFFECTS OF BREAKOUT FORCE AND BACKLASH**



control surface will return to the trimmed position after the cockpit control has been displaced and released. For a reversible system the primary centering tendency comes from the aerodynamic hinge moments. For an irreversible system a feel spring would have the same effect. Good centering aids the pilot in making small momentary corrections from a trimmed flight condition. Sources of trouble occur when the pilot releases the force on the cockpit control, but the control valve does not center, causing the control surface to continue to move. The end result is again to make the pilot work harder to maintain steady flight or to make small corrections since he must now consciously take out each correction.

#### Phase Lag:

In virtually all control systems there is a finite time delay between a cockpit control deflection and the corresponding flight control deflection. In most cases, this time is extremely small. If the flight control system is relatively large and complicated, however, this time lag can become significant. It may be that this control system lag plus the airplane response time could be relatively large. In such cases, if the pilot wished to make a series of rapid control inputs, he might find the aircraft motions out of phase with his desires. Control system lag plus aircraft response time is the phase lag.

Another associated problem which can develop is that the natural frequency of the cockpit control or control system may be near the natural frequency of the associated airplane modes of motion in a portion of the performance envelope. If, for instance, the centering spring is weak, in order to produce a light stick force gradient, the natural frequency of the control system may be low

enough to couple with the short period or Dutch roll mode.

#### Trim:

Trimming is accomplished on virtually all artificial feel systems by shifting the no-load position of the feel device. The trimming device should have enough authority to reduce the control forces to zero throughout the flight envelope. The rate of movement of trim should be determined by the runaway trim condition at  $V_{max}$ .

### **9.7 ARTIFICIAL FEEL SYSTEMS**

In this section, a brief description of some typical artificial feel systems will be given.

Without artificial feel in the irreversible control systems, the stick is free to flop around at will.

#### Simple Spring:

The simplest type of feel system is that in which the cockpit control is restrained from movement in either direction by a linear spring. The control force is then simply proportional to control deflection.

$$\Delta F = K\Delta\delta \quad (9.1)$$

It is worth noting that this type is often called a "bungee." It is unfortunate, but a downspring is also sometimes called a bungee. (In this chapter the term bungee will be used to mean a centering or feel spring.) Although the simple spring does give the pilot some feel of the aircraft response, the stick force versus control surface deflection is a constant regardless of airspeed (ignoring aeroelastic and Mach effects). Therefore, for aircraft with wide speed ranges their use for longitudinal control is unsatisfactory, since

the stick force per "g" is lowest for high speeds and highest for low speeds. However, this type of feel is often used in aileron systems, and thus explains the relatively high roll rates obtained in some of our high performance aircraft at high subsonic airspeeds.

#### q-Feel:

If the feel system can be made a function of dynamic pressure, by use of bellows, a servo device, or any other means, then the control force will be a function of control deflection and dynamic pressure.

$$\Delta F = Kq\Delta\delta \quad .2)$$

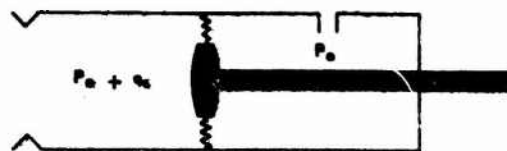
Such a system produces a stick force per g which is invariant with airspeed. Another characteristic of this system is that in regards to speed stability, the stick force varies as a function of "q." Thus "q" feel is an attempt to approximate the feel that is obtained in the reversible system. This device is commonly used for feel in the longitudinal control of many modern aircraft.

There are some problems that reduce the utility of this system somewhat. Since the feel is based on dynamic pressure and not Mach number, the matching of the system to perform correctly at subsonic speeds will result in much larger forces supersonically because nearly twice as large control movements are needed for the same response. Other problems inherent in the "q" system are compressibility and position error effects in the transonic region, however, these can be alleviated to some extent by suitable scheduling, or by a Mach cutoff through a specific range of the transonic region.

One method of mechanizing the  $Kq\Delta\delta$  of feel forces is to produce a force proportional to  $q_c$  by

ram air bellows and piston combination shown in figure 9.13.

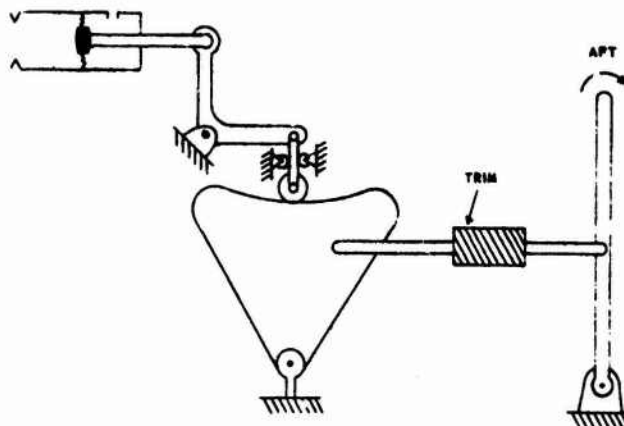
FIGURE 9.13  
RAM AIR BELLOWS AND PISTON



The pressure drop across the piston is the difference between total and static pressure. Hence the force on the piston is approximately the product of the face area and the compressible dynamic pressure.

The  $q_c$  force is fed through a rocking cam to the cockpit control as shown in figure 9.14.

FIGURE 9.14  
RAM AIR FEEL SYSTEM



Ram air provides at all times a force on the cam. As the stick moves aft the cam rotates clockwise and the  $q_c$  force creates a counter-clockwise moment on the cam which must be balanced by a positive stick force.

### V - Feel:

When the feel system is made a function of airspeed, then the control force will be a function of control deflection and airspeed.

$$\Delta F = KV\Delta\delta \quad (9.3)$$

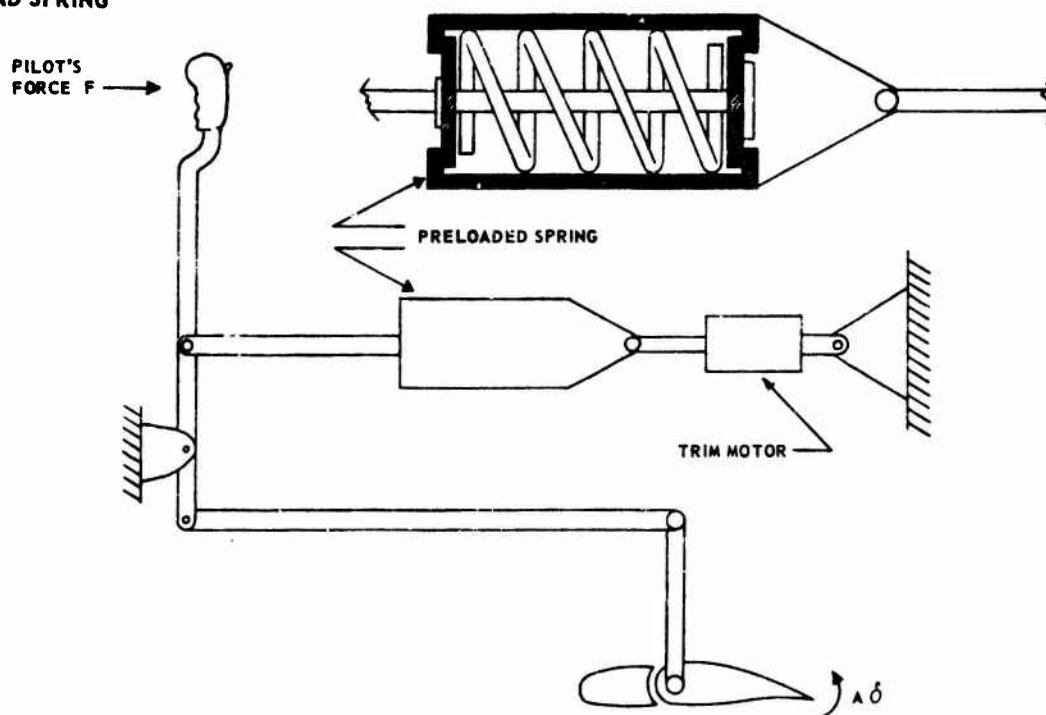
This system, primarily used by the British, is desirable for much the same reason as the  $q$  - feel system, and similarly runs into some of the same difficulties in the transonic regions.

### Preloaded Spring:

If a significant amount of friction exists in a flight con-

trol system, the control centering is likely to be poor. One method of eliminating this problem is to use a preloaded spring on the artificial feel system. A preloaded spring is one which has already been compressed, usually in a cylinder, in such a way that some specified force must be applied to compress it further, as indicated in figure 9.15. Thus, in order for the pilot to move the stick at all, he must supply this force. The magnitude of the preload can be adjusted to just exceed the force required to overcome system friction, thus assuring good centering. The stiffness of the spring can be chosen to give the desired stick force gradient.

FIGURE 9.15  
PRE-LOAD SPRING



If a very light stick force gradient is desired, then the feel characteristics of the infinite stick force gradient through trim produced by the preload may be objectionable. In this case, a

vigorous friction reduction program would probably be a better solution to the centering problem than the preloaded spring.

#### Nonlinear Gearing:

In this arrangement, the ratio of control surface movement to movement of the stick is small near the neutral position changing with displacement to a high ratio near the limits of travel. In this system spring feel is often used with either linear or nonlinear force gradients versus displacement. A typical example of nonlinear gearing is used on the F-100 slab tail in conjunction with a nonlinear spring. The original F-100A with a linear gear, nonlinear spring provided the pilot with oversensitive control at high speed. To remedy this situation, a nonlinear gear was designed. As a result, the amount of stabilizer angle available within one inch forward and aft from the stick neutral position reduced from a total of 7 1/2 to 3 degrees.

There are certain disadvantages inherent in the use of nonlinear gearing. In the longitudinal axis it is desirable that the center of the reduced ratio of control movement occurs at high speed around the trim position. Now if large trim changes are present, say due to cg movement, the trim at high speed would not correspond to the optimum position in the nonlinear slope and the handling characteristics would change. Another problem arises in lightening of force with increasing "g." This is ordinarily solvable with nonlinearity in the feel springs.

#### Variable Gearing:

In a variable gear ratio control the ratio of pilot's control movement to surface movement is altered in flight to fit the desired response characteristics in a given flight regime. This principle of control is applied in the B-58.

The Hustler uses a variable gear system which is varied automatically with dynamic pressure and g loading.

#### Stability Augmentation:

It is frequently necessary to augment the natural stability of an aircraft by synthetic means. This may be because the damping of some mode is too low, or because of actual instabilities, static or dynamic, that are present in some flight regime.

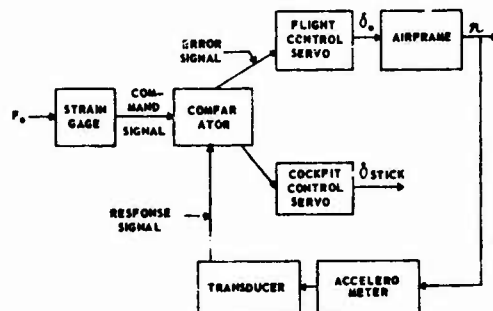
One type of a control damper produces a control force requirement which is proportional to the rate of deflection of the cockpit control or control surface. Usually this damper is a piston which must be moved in a cylinder which contains oil which must be forced through orifices. One possible use of a damper might be to prevent pilots "beating the bobweight." One difficulty in installing such a device is that this damper might also prevent the pilot from making necessary rapid corrections when needed; say, upon landing the aircraft.

From the aerodynamic standpoint, stability augmentation consists of altering one or more of the stability derivatives by automatically displacing one or more of the control surfaces in response to motion of the airframe. For example, if the aircraft is statically unstable longitudinally ( $C_{m\alpha} > 0$ ), then it could in principle be stabilized by sensing  $\alpha$ , and producing an elevator deflection proportional and opposite to it; i.e., let  $\delta_e = k\alpha$ , then

$$\Delta C_{m\alpha} = - \frac{\partial C_m}{\partial \delta_e} \frac{\partial \delta_e}{\partial \alpha} = - K C_{m\delta_e} \quad (9.4)$$

A common example of the stability augmentor is the yaw damper. In its simplest form, it increases the damping in the derivative  $C_{nr}$ , but can also be designed to alter  $C_{ng}$  and  $C_{np}$ .

FIGURE 9.16  
SCHEMATIC OF A RESPONSE FEEL SYSTEM



Other names occasionally used for this system are "Synthetic Feel" or "Stick Steering."

If the desired response (say, g's) to a cockpit input (usually force) is sensed, and if through an appropriate servo device the flight control surface deflection is varied in order to correct the output to the desired response, the pilot has "response feel."

$$F = KX \text{ (desired response)} \quad (9.5)$$

For example, suppose the longitudinal control system of an aircraft worked as follows (figure 9.16):

1. The pilot exerts a force on the stick.
2. The stick force is converted to a voltage - the signal.
3. This command signal voltage is fed to a comparator, then to a flight control position servo, and also to a cockpit control position-servo.
4. These position-servos convert the signals to valve movements, which direct fluid to the appropriate side of pistons, causing flight control and stick movement. (Thus the name "stick steering." The pilot thinks he is moving the stick, but really it is being

driven by the cockpit control position-servo.)

5. The flight control movement produces a desired aircraft response, say normal acceleration.
6. This response is sensed by an accelerometer and converted into a voltage - the "response signal."
7. This response signal voltage is sent to the comparator where command and response signals are continually compared. The difference is called error signal. The comparator must be given the desired stick force per g in order to compare the command response signals. Thus the comparator is the origin and heart of the control system.
8. The error signal in the comparator causes further servo and control motion until the response and command are equal. Then the error signal is zero and no further control motion takes place until the pilot demands it by changing stick force.

As a byproduct of this type of control system, the aircraft which might otherwise have undesirable characteristics can be stabilized by introduction of a damping loop in the circuitry. In this case a rate gyro (sensing pitch rate in longitudinal oscillations in poorly damped flight regions) could be made to operate a fast response servo to deflect the elevator in a stabilizing sense, without the pilot being aware of the corrections being made.

## 9.9 ARTIFICIAL FEEL SYSTEM RESPONSE

Typical response for a number of the feel systems just described are shown in the following figures, where

S = Spring Feel

V = V - Feel

q = q - Feel

R = Response Feel

FIGURE 9.17  
STICK FORCE PER "g"

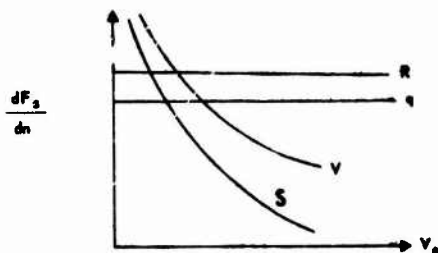


FIGURE 9.18  
OUT-OF-TRIM STICK FORCE

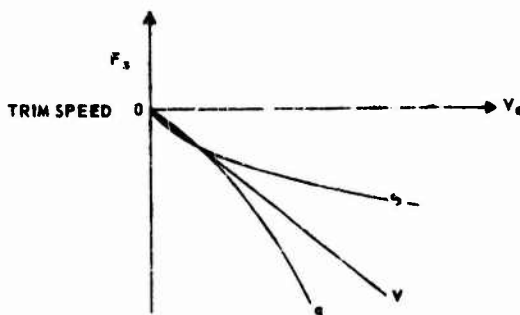


FIGURE 9.19  
AILERON FORCE PER UNIT ROLL RATE

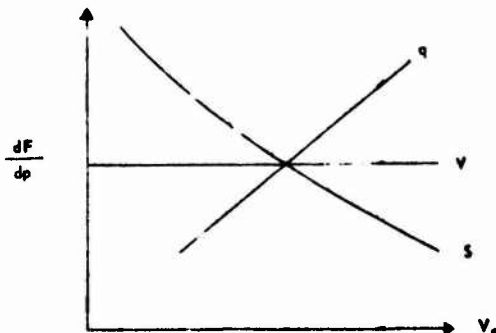
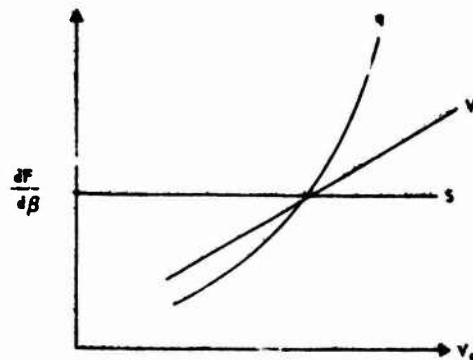


FIGURE 9.20  
RUDDER FORCE PER UNIT SIDESLIP ANGLE

(Assuming  $d\delta_z/d\beta = K$ )



The stick force per g plot (figure 9.17) can be analyzed in the following way:

$$\frac{d\delta_e}{dn} = \frac{K_1}{V^2} \quad \text{For all feel systems except the response feel system} \quad (9.6)$$

$$\frac{dF_s}{d\delta_e} = K_2 \quad \text{For spring feel} \quad (9.7)$$

$$\frac{dF_s}{d\delta_e} = K_3 V \quad \text{For V - Feel} \quad (9.8)$$

$$\frac{dF_s}{d\delta_e} = K_4 V^2 \quad \text{For q - Feel} \quad (9.9)$$

Since  $d\delta_e/dn$  is not a function of  $n$ , the derivatives can be multiplied

$$\frac{dF_s}{dn} = \frac{dF_s}{d\delta_e} \frac{d\delta_e}{dn} \quad (9.10)$$

$$\frac{dF_s}{dn} = \frac{K_1 K_2}{V^2} = f\left(\frac{1}{V^2}\right) \quad \text{for spring feel} \quad (9.11)$$

and

$$\frac{dF_s}{dn} = \frac{K_1 K_3}{V} = f\left(\frac{1}{V}\right) \quad \text{for V - feel} \quad (9.12)$$

and

$$\frac{dF_s}{dn} = K_1 K_4 = \text{constant for q - feel} \quad (9.13)$$

For the response feel  $dF_s/dn = K_5$ . Where  $K_5$  is the desired stick-force per g which is built into the comparator.

The reader may find it worthwhile to analyze the other response curves in a similar manner. Warning: Since  $d\delta_e/dV$  is a function of velocity the method of multiplying will not work. A little reflection should convince the reader that a response feel system that senses a parameter such as normal acceleration or rolling acceleration will require no force by the pilot to maintain a steady state change of airspeed or roll rate. The shape of the response curves for a response feel system would depend entirely on what was built into the comparator or what additional feel device was added.

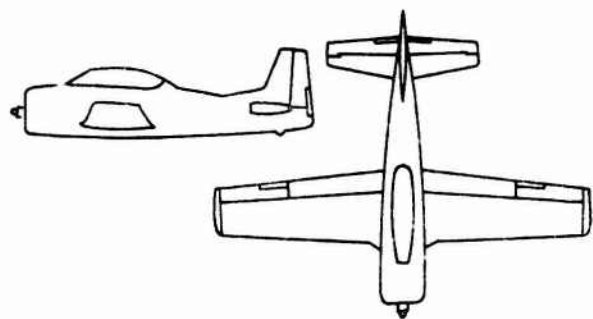
A glance at the stick force per g plot shows that the spring and q-feel curves are quite different. The q - spring gives the pilot a constant stick-force per g throughout the aircraft speed range. This constant measure of response is generally considered to be superior to the variable response encountered when the feel is provided by a simple linear spring. In the latter case, the stick force gradient might well be too high in the pattern and/or dangerously low at high speeds where the aircraft is usually capable of generating its limit load factor. The advantage of the simple spring is its simplicity and reliability. The designer must decide what degree of sophistication, and therefore, complexity must go into the system. The test pilot must then pass judgment on his decision.

## 9.9 CONTROL SYSTEM EXAMPLES

The previous paragraphs have discussed the different classifications of control systems and how they are used to give the pilot control of his aircraft. It is of interest now to look at some of

the actual control systems of a number of aircraft and observe how the different designers have made use of the various types of controls to give the pilot a desirable control system. Descriptive summaries of the control systems of fourteen different aircraft are given in the following paragraphs. There are other aircraft that might have been included such as the F-100, F-106, B-66, etc. However, their control systems were so similar to one or more of the systems given that they were excluded. The reader might consider some of the aircraft described in this section rather outmoded, however, this selection includes the various types of control systems. Further, it is wise to consider the possibility of future development of subsonic aircraft to be employed in small brush-fire type wars, where the use of the more simple control systems might be advantageous from a reliability standpoint.

NORTH AMERICAN T-28A



The control system of the T-28A is a simple reversible type that consists of conventional elevators, ailerons, and a rudder.

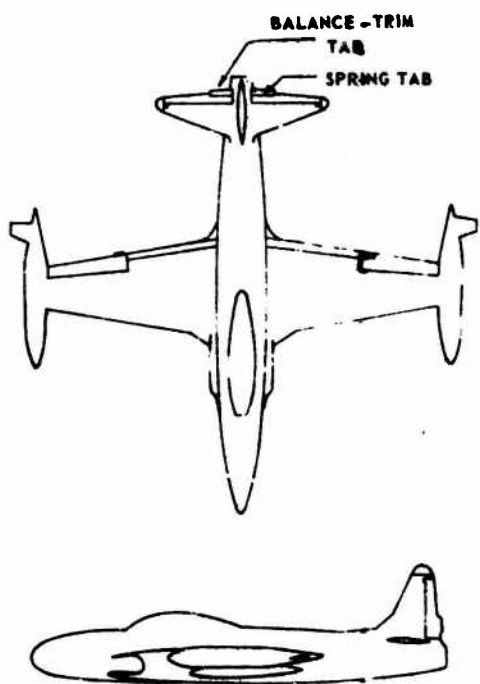
The elevators use aerodynamic balancing in the form of an overhung balance. Mass balancing of the elevator is accomplished by means of a negative bobweight located in the bell crank mechanism. This not only counterbalances the elevator, but reduces elevator

stick forces during accelerated flight. Elevator trim is provided by means of a manually operated trim tab on each elevator.

Aerodynamic balancing is incorporated in the aileron by means of a simple internal seal. Trim is provided by means of a manually-operated trim tab, one on each aileron.

The rudder incorporates aerodynamic balancing by means of a setback hinge. A trim tab located on the trailing edge of the rudder provides directional trim.

#### LOCKHEED T-33A



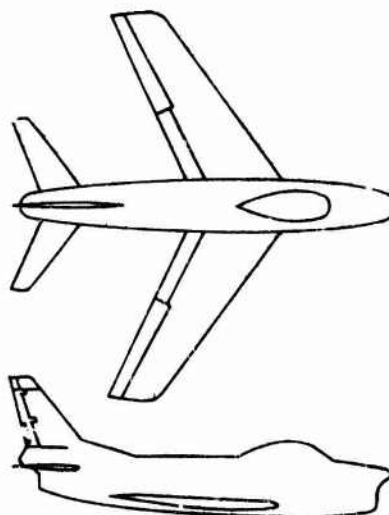
The T-33A has a reversible control system that includes several modifications for improving stick forces or pilot feel.

Longitudinal control is provided by conventional elevators. Aerodynamic balancing is incorporated in the elevators in the form of a

spring tab and a balance tab on each elevator. A small shielded horn balance mounted outboard on each elevator is used for mass balancing of the elevators. Elevator unbalance plus a downspring are used in the control system to increase stick free stability. Trim is provided by an electric actuator that moves the balance tab.

Lateral control is provided by conventional ailerons that are hydraulically boosted. This boost is necessary since the ailerons are not aerodynamically balanced and stick forces are considerable without the hydraulic boost. Lateral trim is provided by an electrically-actuated tab located on the trailing edge of the left aileron.

#### NORTH AMERICAN F-86F



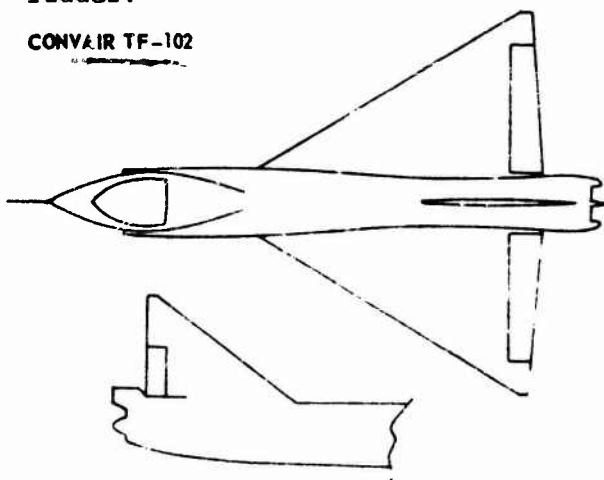
The lateral and longitudinal control systems of the F-86F are irreversible. The directional control system is reversible. The longitudinal control consists of a hydraulically-operated elevator and horizontal stabilizer that are interconnected and operate as one unit. Artificial feel is provided by means of a downspring and a bobweight. The main purpose of

the bobweight is to increase the gradient of stick force versus load factor. Longitudinal trim is accomplished by repositioning the downspring by an electric actuator. This, in effect, changes the neutral (no load) position of the control stick.

Lateral control is provided by hydraulically operated ailerons. Artificial feel is provided by a bungee. Lateral trim is accomplished in the same manner as longitudinal trim.

Directional control is provided by a mechanically-operated rudder. Rudder trim is accomplished by an electrically-actuated tab on the rudder. Mass balancing is incorporated in the rudder by the addition of weight in a small horn balance in the top part of the rudder.

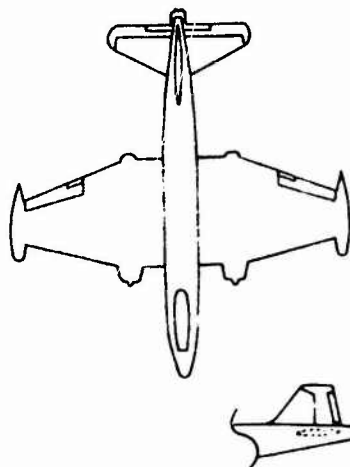
CONVAIR TF-102



The TF-102 has a hydraulically-operated irreversible control system. Being a delta wing aircraft, the TF-102 utilizes elevons instead of ailerons and elevator control surfaces. The elevons when moved coincidentally act as elevators and when moved differentially act as ailerons. This is accomplished by a mixer assembly which consists of a bell crank assembly which rotates to provide aileron action

and moves fore and aft to provide elevator action. A rudder mounted on the vertical stabilizer is used for directional control. Aileron artificial feel is provided the pilot by means of a bungee. Elevator feel is provided by a bungee, a bobweight, and a variable feel force cylinder that incorporates ram air pressure. This is better known as a "Q-feel" system. The bobweight is not only used to increase the gradient of stick force versus load factor, but to counter the negative bobweight effect of the control column. Rudder feel is provided by the same method as the elevator. Trim is accomplished by repositioning the neutral (no load) positions of the control stick and rudder pedals. In addition, an automatic trim servo is used in the control system to compensate for unstable stick force gradients in the transonic speed region.

MARTIN B-57E



The longitudinal control system is a reversible type and consists of mechanically-operated conventional elevators. Aerodynamic balancing is used on the elevators in the form of a spring tab, unshielded horn balance, and trailing edge strips on each elevator. Mass

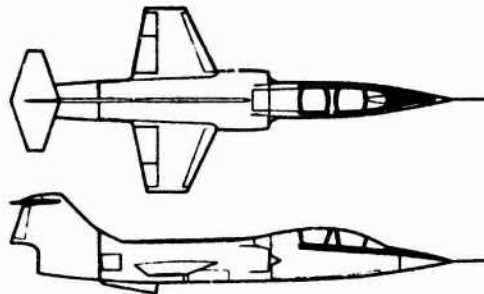
balancing of the elevator is included in the horn balances. The trailing edge strips are used on the elevator to produce a favorable response effect. Longitudinal trim is provided by an electric actuator that changes the angle of incidence of the horizontal stabilizer.

The lateral control system consists of mechanically operated conventional ailerons. A spring tab mounted on each aileron is used for aerodynamic balancing. Lateral trim is provided by an electric actuator that, in effect, shifts the neutral position of the control wheel by varying the elongation of a spring at the base of the control column.

Directional control is provided by a conventional rudder that is normally operated by hydraulic pressure. In the event of hydraulic pressure failure, the system becomes reversible and the rudder can be operated manually. The rudder is balanced aerodynamically by means of a balance tab, unshielded horn balance, and a beveled trailing edge. The rudder is mass balanced by addition of weight in the horn balance.

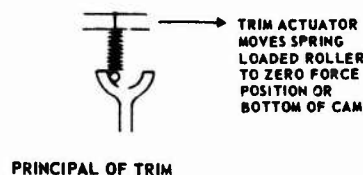
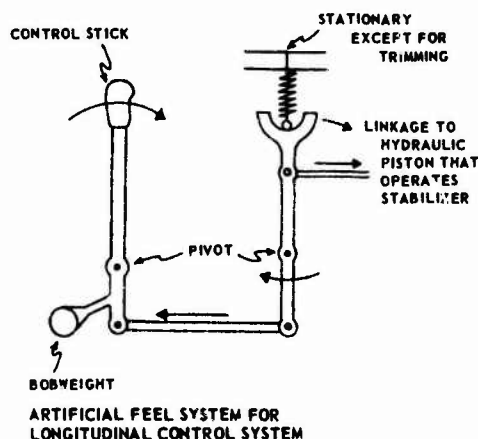
When the rudder is operated by hydraulic pressure, the system becomes irreversible and requires artificial feel. The artificial feel used is a V-feel system which varies the mechanical advantage of the rudder in proportion to the speed of the aircraft, producing high rudder pedal forces at high speeds and low rudder pedal forces at low speeds. When the rudder is operated manually, trim is provided by using the balance tab as a combined trim and balance tab. When the rudder is hydraulically operated, the balance tab is used only as a balance tab and trim is provided by an actuator in the feel system which varies the neutral position of the rudder pedals.

#### LOCKHEED F-104B



The complete control system, i.e., longitudinal, lateral, and directional, of the F-104B is a full hydraulically powered irreversible system.

Longitudinal control is provided by means of a controllable horizontal stabilizer mounted at the top of the vertical stabilizer. Artificial feel is provided by means of a bobweight and a variable feel cam arrangement, which is similar to a bungee or spring feel system. Shown below is a schematic of the cam and spring arrangement which indicates its principle of operation.

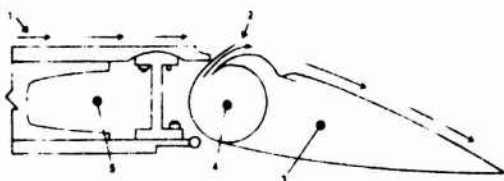


This type of system is unique in that the shape of the curve of stick force versus stabilizer deflection depends on the slope of the cam. Longitudinal trim is provided by an electric actuator that returns the spring loaded roller on the cam to the zero force position. (See preceding sketch.)

Lateral control consists of conventional ailerons mounted on the outboard section of the wing. Artificial feel is provided by means of a bungee. To prevent a dangerously high rate of roll and possible inertial coupling problems at higher speeds the aileron deflections are limited to plus or minus 15 degrees with landing flaps up. This is 5 degrees less deflection than with landing flaps down. Trim is provided by means of an electric actuator that repositions the neutral position of the bungee.

Directional control is provided by a conventional rudder. Artificial feel is provided by means of a bungee. With landing flaps up maximum rudder deflection is decreased from plus or minus 20 degrees to plus or minus 6 degrees by mechanical stops at the rudder pedals. This is done to prevent structural failure of the vertical tail at high speeds. Rudder trim is provided through the yaw damper circuit to the rudder.

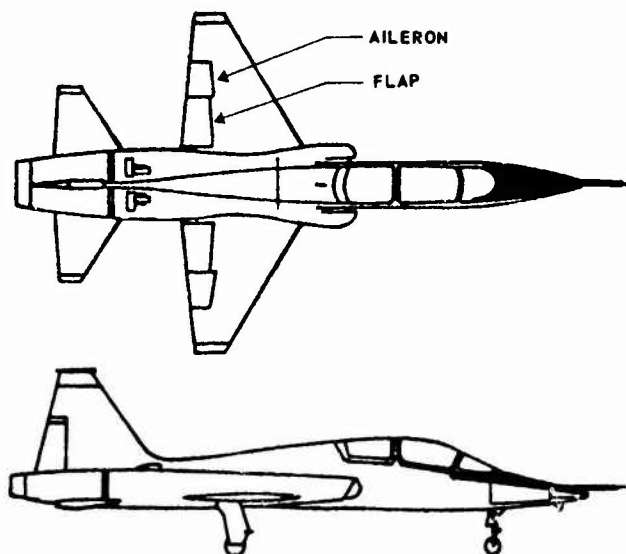
#### BOUNDARY LAYER CONTROL



1. Airflow
2. Boundary layer nozzle
3. Trailing edge flap
4. Boundary layer control duct
5. Aft wing section

In the above figure the air is bled from the compressor and ducted to the boundary layer control manifold which is located above the trailing edge flap hinge. The boundary layer control manifold has a series of nozzles which direct high pressure air over the upper surface of the flap when they are placed in the landing position. The high velocity created by this jet of air re-energizes the boundary layer, causing it to adhere to the curved fairing and bend around and pass over the upper surface of the flap. This induces the adjacent layer of air to adhere and bend through the flap deflection angle, thus preventing airflow separation and resulting in a reduced landing speed. The system operation is automatic.

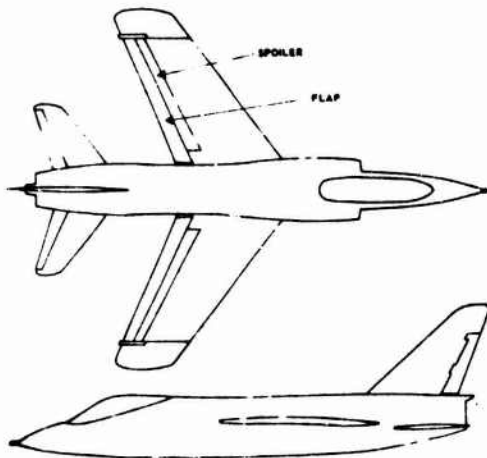
#### NORTHROP T-38



The T-38 has a fully powered irreversible control system with conventional ailerons and rudder, and an all-moveable horizontal tail. This system is quite similar to the F-104B except for the type of artificial feel in the horizontal stabilizer system. All control surfaces are hydraulically operated. The horizontal tail system's artificial

feel consists of a bungee and a positive bobweight. Artificial feel for the ailerons and rudder is achieved by means of bungees. Maximum rudder deflection is reduced also with the landing gear up to prevent structural damage to the vertical tail. Trim is provided by electric actuators that reposition the neutral (no load) position of the bungees. The flaps are mechanically interconnected to the horizontal tail to automatically change its angle to the trim position when the flaps are actuated - trailing edge down when the flaps are extended.

#### GRUMMAN F-11F (TIGER CAT)



The longitudinal control system of the F-11F is a hydraulically-operated irreversible system consisting of an all-movable horizontal stabilizer and geared elevator much like that of the F-86F. One unique feature of the system is that when the wing flap control is in the "up" position, the elevator is automatically locked in line with the stabilizer and the two units move as a single unit. When the wing flaps are in the down position the elevator is geared to stabilizer motion. When the control stick is pushed forward the leading edge of the stabilizer moves up and the trailing edge of the elevator moves down. When the control stick is pulled aft, the stabilizer leading

edge moves down and the elevator trailing edge moves up. This, in effect, changes the camber of the tail as well as the angle of attack which improves pitch control in the lower speed range.

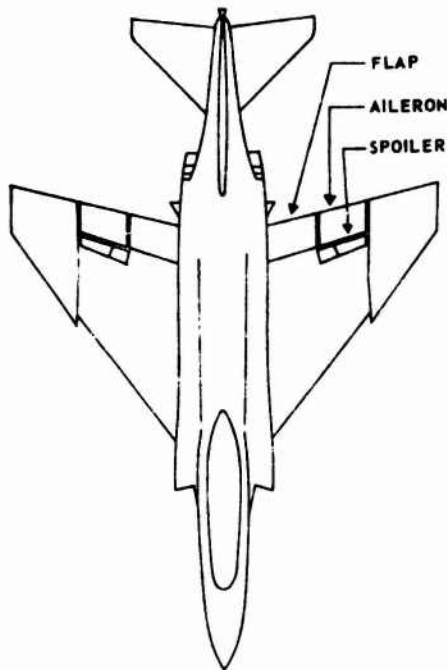
Artificial feel is supplied to the pilot by means of a cam and spring-loaded follower (modified bungee) that is quite similar to that of the F-104B. (See schematic of control system.) In addition, a positive bobweight is used in the system for feel. Longitudinal trim is provided by an electric actuator that varies the neutral position of the artificial feel system which in turn varies the neutral position of the stick.

The lateral control system is a hydraulically-operated irreversible system consisting of a movable flaperon (spoiler) on the top of each wing. This type of system is used on the F-11F so that full span wing flaps can be used to provide lower carrier approach speeds. The flaperons are hinged forward and move through an arc of 55 degrees when the control stick is deflected fully. When control stick is deflected right, the right flaperon rises with the left flaperon remaining flush and vice-versa. The artificial feel system consists of a cam and spring-loaded follower (modified bungee) that is almost identical to the longitudinal system arrangement. Lateral trim is provided by the same method as longitudinal trim, i.e., varying the neutral position of the artificial feel system.

The directional control system is irreversible and consists of a hydraulically-operated conventional rudder. To prevent possible structural failure of the vertical tail the rudder is limited to 5 degrees either side of neutral by automatic rudder pedal position stops with flaps in up position. With flaps in the down position rudder travel is increased to 22

degrees either side of neutral for increased directional control at lower speeds. This is quite similar to the system of the F-104B. The artificial feel system consists of a cam and spring-loaded follower (modified bungee) quite similar to that of the longitudinal and lateral system. Movement of the cam against the follower induces a force in the control system that opposes pedal movement. This force is dependent on pedal position only. Trim is provided by an electric actuator that operates through the yaw damper to trim the rudder in much the same way as the F-104B rudder trim.

#### MCDONNELL F-4C



The longitudinal control system of the F-4C consists of a hydraulically-operated horizontal stabilizer referred to by McDonnell as a "stabilator." Artificial feel is provided the system by the addition of a bobweight and a Q-spring which makes stick force a function of dynamic pressure. Trim is accomplished by adjusting the neutral

position of the artificial feel system.

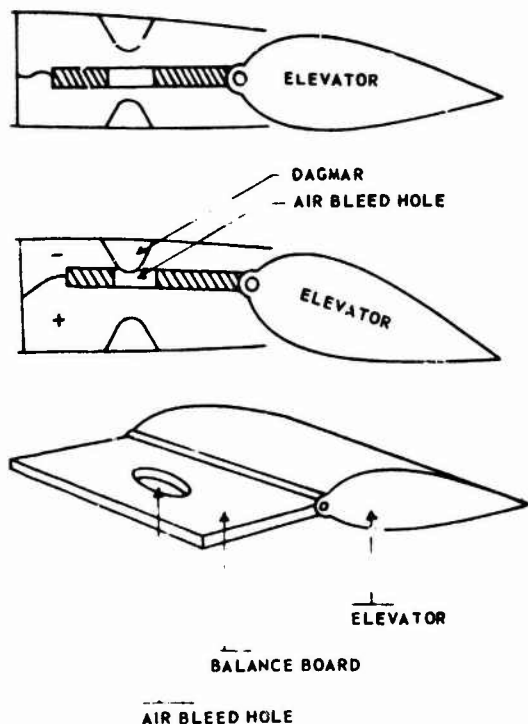
The lateral control system consists of hydraulically-operated ailerons and spoilers. The system is quite unique in that the ailerons deflect downward only. When the control stick is deflected right for a roll to the right, the right aileron remains flush, the right spoiler deflects upward, the left aileron deflects downward, and the left spoiler remains flush. The system operates opposite for a roll to the left. Spoilers are used in the F-4C to partially alleviate wing twisting during high-speed roll maneuvers, and the aileron spoiler combination is used to increase roll rate and roll effectiveness at low speeds. Maximum roll rate is decreased above 220 knots indicated airspeed by a pressure switch in the system that decreases the maximum available hydraulic pressure which in turn reduces the deflection of the aileron and spoilers in proportion with dynamic pressure. The reason for this is to avoid dangerously high rates of roll which might cause inertial coupling. Artificial feel is provided the system by means of a bungee. Trim is made available by adjusting the neutral position of the bungee.

An aileron-rudder interconnect is incorporated in the lateral and directional control system to automatically coordinate turns made at speeds below 225 knots. This is accomplished by providing rudder displacement as a function of aileron displacement.

#### CONVAIR 880

The longitudinal control system of the 880 is a reversible system that consists of conventional elevators mounted on the rear spar of the horizontal stabilizer. Movement of the elevators is accomplished indirectly by movement of a control or servo tab located on

the trailing edge of each elevator. An unusual feature in the system is that the left and right elevators are independently hinged and may be moved independently of each other on the ground. However, the elevators will always act together in flight since the control tabs are interconnected. Centering springs located at the elevator hinge line and attached to the control tabs are used to overcome system friction, and return the control tabs to neutral when the pilot releases stick pressure. An internal seal type of aerodynamic balancing called a balance board is used on the elevators to reduce stick forces. The balance boards are quite unique in their design in that they provide maximum assistance to the pilot at large deflection of the control surfaces where he needs assistance and very little at small deflection of the control surfaces where assistance is not required. The principle of operation of the balance boards is described as follows. (Refer to figure shown below.)



If the control surface is deflected in either direction, a differential pressure will exist between each side of the balance board, with the increased pressure being on the deflected side. This differential pressure will exert an additional force that will aid control surface movement. The differential pressure will increase as the control surface deflection is increased due to the dome-shaped "dagmar" entering the airbleed hole and effectively sealing off each side of the balance board. At maximum deflection of the elevator, the airbleed hole is closed completely and the differential pressure is at its maximum. This gives the maximum aid in moving the control surface at precisely the time it is required. Once the control surface is again centered differential pressures will no longer exist.

Longitudinal trim is normally provided by an electrical actuator that changes the angle of incidence of the horizontal stabilizer. In the event of electrical failure, trim can be accomplished manually by trim cables.

The lateral control system is a partially reversible system that consists of conventional ailerons augmented by two sets of wing spoilers on each wing. The ailerons, located approximately half way out the wing span, between the inboard and outboard flaps (see picture), are indirectly operated through control tabs on the trailing edge of each aileron. The ailerons are aerodynamically balanced by balance boards similar to those of the longitudinal system. The wing spoilers, located forward of each flap are hydraulically operated and incorporate a blowdown safety design that prevents damaging the spoilers at high speeds. The lateral control system operates as follows (right roll example): When the control wheel is rotated to the right, the right aileron and spoilers deflect up, the left

aileron deflects down, and the left spoilers remain flush. The spoilers are also used as speed brakes by deflection of both sets of wing spoilers together. If the spoilers are being used as speed brakes, and the control wheel is rotated right, the right spoiler will remain in the up position and the left spoiler will deflect downward toward the flush position. Lateral trim is provided by means of a manually-operated trim tab located on the trailing edge of each aileron.

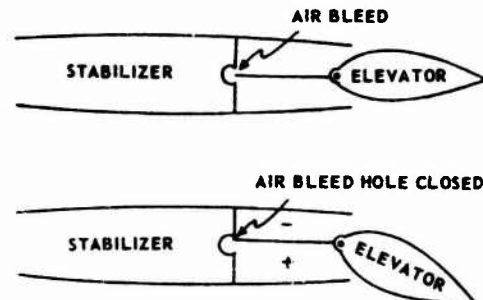
The directional control system is reversible and consists of a conventional rudder that is operated indirectly by a control tab on the trailing edge of the rudder. A center spring mechanism is used in conjunction with the control tab to return it to its neutral position when pressure is removed from the rudder pedals. The rudder is aerodynamically balanced by the use of five balance boards identical to those used in the elevator. One unique feature of the control system is that although it is a reversible system, a feel system is designed into the rudder control system. This makes the system force limited at high airspeeds, thus preventing overstressing of the vertical tail through excessive side loads. Rudder trim is provided by means of a manually-operated trim tab located on the trailing edge of the rudder.

#### BOEING KC-135

The longitudinal control system is a reversible system and control is provided by means of elevators mounted on an adjustable horizontal stabilizer. The elevators are independently hinged in much the same way as those of the Convair 880. The elevators are deflected together by manually-operated interconnected control tabs (servo tabs) located on the trailing edge of the elevators. Once full control tab is attained, additional elevator deflection can be obtained by further movement of

the control column, though this is not practical during flight due to high stick forces. Each elevator is aerodynamically balanced by means of five balance panels (internal seals) that operate quite similarly to the balance boards of the Convair 880.

Shown below is a sketch of the balance panel.



Trim is accomplished by varying the angle of incidence of the horizontal stabilizer by an electrical or manual actuator. In addition, a trim tab is located on each elevator and is actuated by movement of the horizontal stabilizer. The purpose of these tabs is to position the elevators in line with the stabilizer and reduce the upward and downward movement of the elevator which is a result of aerodynamic loads on the elevator caused by positioning of the stabilizer.

The lateral control system is a partially reversible system and consists of inboard and outboard ailerons that are used in conjunction with two sets of spoilers on the top of each wing. The inboard ailerons, which are considerably smaller than the outboard ailerons are used in conjunction with the spoilers for lateral control throughout the speed range of the aircraft. When flaps are lowered for low speed flight, the outboard ailerons are locked out of the system and remain faired in the neutral position. This prevents wing twist at high speeds. The outboard ailerons

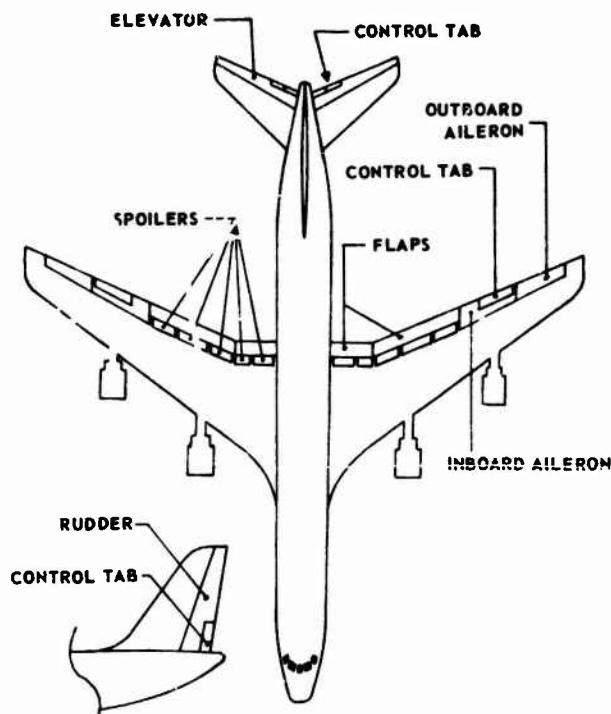
are aerodynamically balanced by use of internal seals and a spring tab on each aileron. Movement of the inboard ailerons is by action through the spring tab. The spring tab is also used as a trim tab for lateral trim. The lateral control system operates as follows (right roll example): When the control wheel is rotated to the right, the right ailerons and right spoilers deflect upward. The left ailerons deflect downward, and the left spoilers remain flush. The spoilers on both wings are also used as speedbrakes by being deflected together. If the spoilers are being used as speed brakes, and the control wheel is rotated right, the right spoiler will remain up, and the left spoiler will deflect down.

The directional control system of the KC-135 consists of a conventional rudder that incorporates four rudder tabs, i.e., a trim tab, a spring tab, a control tab, and an antibalance tab. The antibalance tab is located on the trailing edge of the control tab. The purpose of the antibalance tab is to increase the effectiveness of the control tab. In addition five balance panels (internal seal) are used for aerodynamic balancing of the rudder. The rudder control surface is mechanically operated from the pilot's rudder pedal through the control tab and spring tab. Rudder trim is provided by a mechanical manually-operated trim tab.

The control system of the Boeing 707 is almost identical to the KC-135 except for the directional control system. The 707's rudder incorporates aerodynamic balancing only in the form of balance panels. The rudder has one control tab used to deflect the rudder. Whenever the rudder is deflected in excess of 15 degrees, the control tab hits its stop, a rudder hydraulic control unit takes over the rudder system and it then

becomes an irreversible system. Artificial feel is supplied to the pilot by a Q-spring assembly which provides an artificial feel proportional to the dynamic pressure and rudder deflection.

#### DC-8



The longitudinal control system of the DC-8 is a partially reversible system that consists of conventional elevators that are manually operated indirectly by control tabs. Aerodynamic balancing is incorporated in the elevator by means of a balance tab on each elevator and an overhang balance. A centering spring is provided in the system to provide more positive centering of the controls and also for additional control forces. Trim is provided by means of an electric actuator that varies the angle of incidence of the horizontal stabilizer.

The lateral control system consists of inboard and outboard ailerons interconnected by a torsion rod. Wing spoilers on the top of

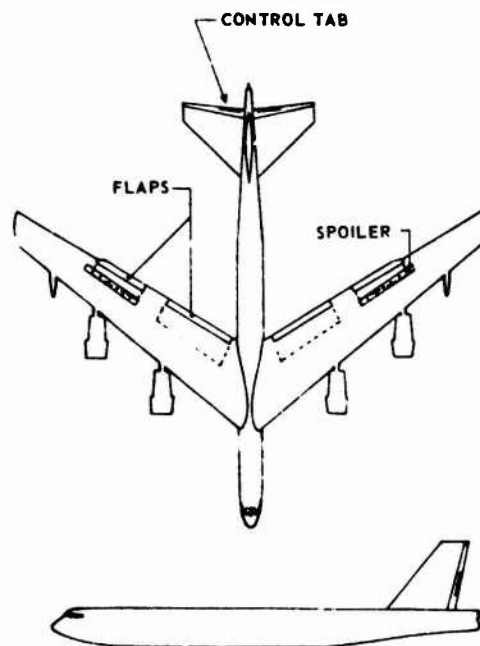
each wing are used in conjunction with the ailerons, whenever the landing gear is extended, for better lateral control during landing. Both inboard and outboard ailerons use aerodynamic balancing. Hydraulic power normally operates the inboard ailerons which in turn operate the outboard ailerons through the interconnecting torsion rods. At low speeds when the inboard ailerons are deflected, the outboard ailerons deflect an equal amount; however, as speed is increased, the outboard aileron deflection becomes less due to the twisting of the interconnecting torsion rods. This, in effect, reduces wing twist as speed is increased. Artificial feel is supplied to the pilot by means of a bungee spring. In the event that hydraulic power is not available, the inboard ailerons can be manually operated indirectly by control tabs located on the trailing edge of each aileron. These control tabs are normally locked in the neutral position whenever the ailerons are operated by hydraulic power, but automatically unlock when hydraulic power is not available. Lateral trim is accomplished manually by adjusting the neutral (no load) position of the bungee spring.

Directional control is provided by means of a hydraulically-powered conventional rudder. The rudder uses aerodynamic balancing in the form of an overhang balance. Artificial feel is supplied to the system by a bungee spring. When hydraulic power is not available, the rudder can be controlled indirectly by a mechanically-operated control tab. Like the aileron control tab, the rudder control tab is normally locked in the neutral position when hydraulic power is available, but automatically unlocks if hydraulic power is not available. Rudder trim is provided by adjusting the neutral (no load) position of the bungee spring.

#### B-52G

The longitudinal control system of the B-52G is quite similar to that of the 707. The system is reversible and consists of conventional independently hinged elevators positioned by manually-operated interconnected control tabs. The elevator is mass balanced by weights in the elevator forward of its hinge line. Aerodynamic balancing is incorporated in the elevator by use of balance panels (internal seals) which operate similarly to those used in the 707. Additional feel is supplied to the elevator by means of a Q-spring. In addition, a centering spring is used to assist in centering the elevator at low indicated airspeed. Longitudinal trim is provided by an electric actuator that varies the angle of incidence of the horizontal stabilizer.

#### BOEING B-52



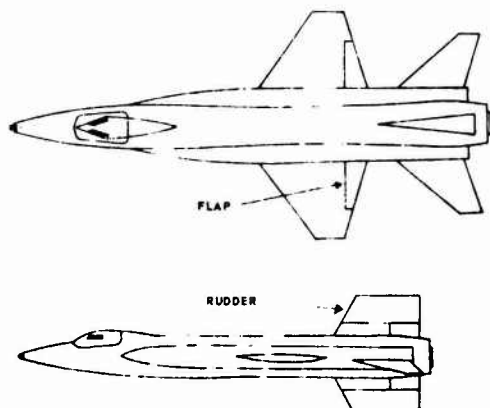
The lateral control system of the B-52G is quite different from similar type aircraft such as the 880 and DC-8 in that no conventional ailerons are used. Lateral con-

control is provided by means of two sets of spoilers located on the top of each wing (see picture). The spoilers are hydraulically operated and artificial feel is supplied to the pilot by means of bungee springs. The spoilers can be deflected fully up to an indicated airspeed of 250 knots. Above this speed, hydraulic pressure is insufficient to obtain full deflection. The reason for this is to reduce wing twist at high indicated airspeeds. The spoilers are also used as speed brakes in the same manner as the 707.

Lateral trim is provided by an electric actuator that repositions the neutral (no load) position of the bungee springs.

The directional control system of the B-52G is reversible and consists of a conventional rudder operated by a control tab on the trailing edge of the rudder. The rudder is balanced aerodynamically by the same method as the elevator, i.e., balance panels. Additional feel is supplied to the pilot by a Q-spring which is almost identical to that of the elevator system. Directional trim is provided by manually adjusting the control tab through the Q-spring feel system.

X-15



The X-15 has two control systems, an aerodynamic flight-control system and a reaction control system. The reaction control system, often called a ballistic control system, is used to control the aircraft's attitude and altitude where the aerodynamic surfaces are relatively ineffective.

### 1. Aerodynamic Flight Control System

Longitudinal (pitch) and lateral (roll) control in this system is provided by a hydraulically-actuated horizontal stabilizer. The stabilizer consists of two all-movable, one-piece surfaces that can be moved simultaneously or differentially. Longitudinal (pitch) control is obtained by simultaneous movement of the left and right stabilizers. Lateral (roll) control is obtained by differential movement of the horizontal stabilizer surfaces. Combined pitch and roll control is obtained by compound movement of the horizontal stabilizer surfaces. Artificial feel is supplied to the system by bungees. Longitudinal trim is obtained by shifting the neutral (no load) position of the bungee. Lateral trim is adjustable only on the ground.

Directional control is obtained by deflection of the upper and lower stabilizers that are interconnected and hydraulically actuated. Artificial feel is provided by a bungee. Prior to landing, the lower stabilizer is dropped since it extends below the landing skids. Directional trim is adjustable only on the ground.

### 2. Reaction Control System

Reaction control is provided by small rockets located in the nose section and wing that use a monopropellant (hydrogen peroxide) which is converted by catalytic action to superheated steam and oxygen. The

reaction of the escaping gases causes the aircraft to move about the selected or combined axes. The control used by the pilot consists of a control stick handle located on the left console of the cockpit. Upward and downward movement of the control handle operates the rockets located on the top and bottom of the nose section, respectively, and gives the aircraft pitch control. Sideward movement of the control handle operates the rockets located on the sides of the nose section and provides the aircraft with directional control. Rotating the control handle operates the rockets located in the

wing section and provides lateral control.

Artificial feel is provided the pilot for all three axes of operation by spring bungees connected to the system. The angular acceleration and velocity of the aircraft vary with the amount and duration of the ballistic control handle application. The velocity tends to sustain itself after the stick is returned to the neutral position. A subsequent stick movement opposite to the initial one is required to cancel the original induced velocity.